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 Branch of mathematics For the kind of algebraic structure, see Algebra over a field. For other uses, see Algebra (disambiguation). Elementary algebra studies which values solve equations formed using arithmetical operations.Abstract algebra studies algebraic structures, such as the ring of integers given by the set of integers (Z) {\displaystyle (\mathbb {Z})} together with operations of addition (+ {\displaystyle +}) and multiplication (× {\displaystyle \times }). Algebra is a branch of mathematics that deals with abstract systems, known as algebraic structures, and the manipulation of expressions within those systems. It is a generalization of arithmetic that introduces variables and algebraic operations other than the standard arithmetic operations, such as addition and multiplication. Elementary algebra is the main form of algebra taught in schools. It examines mathematical statements using variables for unspecified values and seeks to determine for which values the statements are true. To do so, it uses different methods of transforming equations to isolate variables. Linear algebra is a closely related field that investigates linear equations and combinations of them called systems of linear equations. It provides methods to find the values that solve all equations in the system at the same time, and to study the set of these solutions. Abstract algebra studies algebraic structures, which consist of a set of mathematical objects together with one or several operations defined on that set. It is a generalization of elementary and linear algebra since it allows mathematical objects other than numbers and non-arithmetic operations. It distinguishes between different types of algebraic structures, such as groups, rings, and fields, based on the number of operations they use and the laws they follow, called axioms. Universal algebra and category theory provide general frameworks to investigate abstract patterns that characterize different classes of algebraic structures. Algebraic methods were first studied in the ancient period to solve specific problems in fields like geometry. Subsequent mathematicians examined general techniques to solve equations independent of their specific applications. They described equations and their solutions using words and abbreviations until the 16th and 17th centuries when a rigorous symbolic formalism was developed. In the mid-19th century, the scope of algebra broadened beyond a theory of equations to cover diverse types of algebraic operations and structures. Algebra is relevant to many branches of mathematics, such as geometry, topology, number theory, and calculus, and other fields of inquiry, like logic and the empirical sciences. Algebra is the branch of mathematics that studies algebraic structures and the operations they use.[1] An algebraic structure is a non-empty set of mathematical objects, such as the integers, together with algebraic operations defined on that set, like addition and multiplication.[2][a] Algebra explores the laws, general characteristics, and types of algebraic structures. Within certain algebraic structures, it examines the use of variables in equations and how to manipulate these equations.[4][b] Algebra is often understood as a generalization of arithmetic.[8] Arithmetic studies operations like addition, subtraction, multiplication, and division, in a particular domain of numbers, such as the real numbers.[9] Elementary algebra constitutes the first level of abstraction. Like arithmetic, it restricts itself to specific types of numbers and operations. It generalizes these operations by allowing indefinite quantities in the form of variables in addition to numbers.[10] A higher level of abstraction is found in abstract algebra, which is not limited to a particular domain and examines algebraic structures such as groups and rings. It extends beyond typical arithmetic operations by also covering other types of operations.[11] Universal algebra is still more abstract in that it is not interested in specific algebraic structures but investigates the characteristics of algebraic structures in general.[12] The word algebra comes from the title of al-Khwarizmi's book Al-Jabr.[13] The term "algebra" is sometimes used in a more narrow sense to refer only to elementary algebra or only to abstract algebra.[14] When used as a countable noun, an algebra is a specific type of algebraic structure that involves a vector space equipped with a certain type of binary operation.[15] Depending on the context, "algebra" can also refer to other algebraic structures, like a Lie algebra or an associative algebra.[16] The word algebra comes from the Arabic term الجبر (al-jabr), which originally referred to the surgical treatment of bonesetting. In the 9th century, the term received a mathematical meaning when the Persian mathematician Muhammad ibn Musa al-Khwarizmi employed it to describe a method of solving equations and used it in the title of a treatise on algebra, al-Kitāb al-Mukhtaṣar fī Ḥisāb al-Jabr wal-Muqābalah [The Compendious Book on Calculation by Completion and Balancing] which was translated into Latin as Liber Algebrae et Almucabola.[c] The word entered the English language in the 16th century from Italian, Spanish, and medieval Latin.[18] Initially, its meaning was restricted to the theory of equations, that is, to the art of manipulating polynomial equations in view of solving them. This changed in the 19th century[d] when the scope of algebra broadened to cover the study of diverse types of algebraic operations and structures together with their underlying axioms, the laws they follow.[21] Main article: Elementary algebra Algebraic expression notation: 1 – power (exponent) 2 – coefficient 3 – term 4 – operator 5 – constant term c {\displaystyle c} – constant x {\displaystyle x} y {\displaystyle y} – variables Elementary algebra, also called school algebra, college algebra, and classical algebra,[22] is the oldest and most basic form of algebra. It is a generalization of arithmetic that relies on variables and examines how mathematical statements may be transformed.[23] Arithmetic is the study of numerical operations and investigates how numbers are combined and transformed using the arithmetic operations of addition, subtraction, multiplication, division, exponentiation, extraction of roots, and logarithm. For example, the operation of addition combines two numbers, called the addends, into a third number, called the sum, as in 2 + 5 = 7 {\displaystyle 2+5=7} .[9] Elementary algebra relies on the same operations while allowing variables in addition to regular numbers. Variables are symbols for unspecified or unknown quantities. They make it possible to state relationships for which one does not know the exact values and to express general laws that are true, independent of which numbers are used. For example, the equation 2 × 3 = 3 × 2 {\displaystyle 2\times 3=3\times 2} belongs to arithmetic and expresses an equality only for these specific numbers. By replacing the numbers with variables, it is possible to express a general law that applies to any possible combination of numbers, like the commutative property of multiplication, which is expressed in the equation a × b = b × a {\displaystyle a\times b=b\times a} .[23] Algebraic expressions are formed by using arithmetic operations to combine variables and numbers. By convention, the lowercase letters x {\displaystyle x} , y {\displaystyle y} , and z {\displaystyle z} represent variables. In some cases, subscripts are added to distinguish variables, as in x 1 {\displaystyle x_{1}} , x 2 {\displaystyle x_{2}} , and x 3 {\displaystyle x_{3}} . The lowercase letters a {\displaystyle a} , b {\displaystyle b} , and c {\displaystyle c} are usually used for constants and coefficients.[e] The expression 5 x + 3 {\displaystyle 5x+3} is an algebraic expression created by multiplying the number 5 with the variable x {\displaystyle x} and adding the number 3 to the result. Other examples of algebraic expressions are 32 x y z {\displaystyle 32xyz} and 64 x 1 2 + 7 x 2 − c {\displaystyle 64x_{1}{}^{2}+7x_{2}-c} .[25] Some algebraic expressions take the form of statements that relate two expressions to one another. An equation is a statement formed by comparing two expressions, saying that they are equal. This can be expressed using the equals sign (= {\displaystyle =}), as in 5 x 2 + 6 x = 3 y + 4 {\displaystyle 5x^{2}+6x=3y+4} . Inequations involve a different type of comparison, saying that the two sides are different. This can be expressed using symbols such as the less-than sign (< {\displaystyle {\displaystyle >}), and the inequality sign (≠ {\displaystyle \neq }). Unlike other expressions, statements can be true or false, and their truth value usually depends on the values of the variables. For example, the statement x 2 = 4 {\displaystyle x^{2}=4} is true if x {\displaystyle x} is either 2 or −2 and false otherwise.[26] Equations with variables can be divided into identity equations and conditional equations. Identity equations are true for all values that can be assigned to the variables, such as the equation 2 x + 5 x = 7 x {\displaystyle 2x+5x=7x} . Conditional equations are only true for some values. For example, the equation x + 4 = 9 {\displaystyle x+4=9} is only true if x {\displaystyle x} is 5.[27] The main goal of elementary algebra is to determine the values for which a statement is true. This can be achieved by transforming and manipulating statements according to certain rules. A key principle guiding this process is that whatever operation is applied to one side of an equation also needs to be done to the other side. For example, if one subtracts 5 from the left side of an equation one also needs to subtract 5 from the right side to balance both sides. The goal of these steps is usually to isolate the variable one is interested in on one side, a process known as solving the equation for that variable. For example, the equation x − 7 = 4 {\displaystyle x-7=4} can be solved for x {\displaystyle x} by adding 7 to both sides, which isolates x {\displaystyle x} on the left side and results in the equation x = 11 {\displaystyle x=11} .[28] There are many other techniques used to solve equations. Simplification is employed to replace a complicated expression with an equivalent simpler one. For example, the expression 7 x − 3 x {\displaystyle 7x-3x} can be replaced with the expression 4 x {\displaystyle 4x} since 7 x − 3 x = (7 − 3) x = 4 x {\displaystyle 7x-3x=(7-3)x=4x} by the distributive property.[29] For statements with several variables, substitution is a common technique to replace one variable with an equivalent expression that does not use this variable. For example, if one knows that y = 3 x {\displaystyle y=3x} then one can simplify the expression 7 x y {\displaystyle 7xy} to arrive at 21 x 2 {\displaystyle 21x^{2}} . In a similar way, if one knows the value of one variable one may be able to use it to determine the value of other variables.[30] Algebraic equations can be used to describe geometric figures. All values for x {\displaystyle x} and y {\displaystyle y} that solve the equation are interpreted as points. They are drawn as a red, upward-sloping line in the graph above. Algebraic equations can be interpreted geometrically to describe spatial figures in the form of a graph. To do so, the different variables in the equation are understood as coordinates and the values that solve the equation are interpreted as points of a graph. For example, if x {\displaystyle x} is set to zero in the equation y = 0.5 x − 1 {\displaystyle y=0.5x-1} , then y {\displaystyle y} must be −1 for the equation to be true. This means that the (x , y) {\displaystyle (x,y)} -pair (0 , − 1) {\displaystyle (0,-1)} is part of the graph of the equation. The (x , y) {\displaystyle (x,y)} -pair (0 , 7) {\displaystyle (0,7)} , by contrast, does not solve the equation and is therefore not part of the graph. The graph encompasses the totality of (x , y) {\displaystyle (x,y)} -pairs that solve the equation.[31] Main article: Polynomial A polynomial is an expression consisting of one or more terms that are added or subtracted from each other, like x 4 + 3 x y 2 + 5 x 3 − 1 {\displaystyle x^{4}+3xy^{2}+5x^{3}-1} . Each term is either a constant, a variable, or a product of a constant and variables. Each variable can be raised to a positive integer power. A monomial is a polynomial with one term while two- and three-term polynomials are called binomials and trinomials. The degree of a polynomial is the maximal value (among its terms) of the sum of the exponents of the variables (4 in the above example).[32] Polynomials of degree one are called linear polynomials. Linear algebra studies systems of linear polynomials.[33] A polynomial is said to be univariate or multivariate, depending on whether it uses one or more variables.[34] Factorization is a method used to simplify polynomials, making it easier to analyze them and determine the values for which they evaluate to zero. Factorization consists of rewriting a polynomial as a product of several factors. For example, the polynomial x 2 − 3 x − 10 {\displaystyle x^{2}-3x-10} can be factorized as (x + 2) (x − 5) {\displaystyle (x+2)(x-5)} . The polynomial as a whole is zero if and only if one of its factors is zero, i.e., if x {\displaystyle x} is either −2 or 5.[35] Before the 19th century, much of algebra was devoted to polynomial equations, that is equations obtained by equating a polynomial to zero. The first attempts for solving polynomial equations were to express the solutions in terms of nth roots. The solution of a second-degree polynomial equation of the form a x 2 + b x + c = 0 {\displaystyle ax^{2}+bx+c=0} is given by the quadratic formula[36] x = − b ± b 2 − 4 a c 2 a . {\displaystyle x={\frac {-b\pm {\sqrt {b^{2}-4ac\ }}}{2a}}.} Solutions for the degrees 3 and 4 are given by the cubic and quartic formulas. There are no general solutions for higher degrees, as proven in the 19th century by the Abel–Ruffini theorem.[37] Even when general solutions do not exist, approximate solutions can be found by numerical tools like the Newton–Raphson method.[38] The fundamental theorem of algebra asserts that every univariate polynomial equation of positive degree with real or complex coefficients has at least one complex solution. Consequently, every polynomial of a positive degree can be factorized into linear polynomials. This theorem was proved at the beginning of the 19th century, but this does not close the problem since the theorem does not provide any way for computing the solutions.[39] Main article: Linear algebra Linear algebra starts with the study of systems of linear equations.[40] An equation is linear if it can be expressed in the form a 1 x 1 + a 2 x 2 + . . . + a n x n = b {\displaystyle a_{1}x_{1}+a_{2}x_{2}+...+a_{n}x_{n}=b} , where a 1 {\displaystyle a_{1}} , a 2 {\displaystyle a_{2}} , ..., a n {\displaystyle a_{n}} and b {\displaystyle b} are constants. Examples are x 1 − 7 x 2 + 3 x 3 = 0 {\displaystyle x_{1}-7x_{2}+3x_{3}=0} and 1 4 x − y = 4 {\displaystyle \textstyle {\frac {1}{4}}x-y=4} . A system of linear equations is a set of linear equations for which one is interested in common solutions.[41] Matrices are rectangular arrays of values that have been originally introduced for having a compact and synthetic notation for systems of linear equations.[42] For example, the system of equations 9 x 1 + 3 x 2 − 13 x 3 = 0 2.3 x 1 + 7 x 3 = 9 − 5 x 1 − 17 x 2 = − 3 {\displaystyle {\begin{aligned}9x_{1}+3x_{2}-13x_{3}&=0\\2.3x_{1}+7x_{3}&=9\\-5x_{1}-17x_{2}&=-3\end{aligned}}} can be written as A X = B , {\displaystyle AX=B,} where A {\displaystyle A} , X {\displaystyle X} and B {\displaystyle B} are the matrices A = [9 3 − 13 2.3 0 7 − 5 − 17 0] , X = [x 1 x 2 x 3] , B = [0 9 − 3] . {\displaystyle A={\begin{bmatrix}9&3&-13\\2.3&0&7\\-5&-17&0\end{bmatrix}},\quad X={\begin{bmatrix}x_{1}\\x_{2}\\x_{3}\end{bmatrix}},\quad B={\begin{bmatrix}0\\9\\-3\end{bmatrix}}.} Under some conditions on the number of rows and columns, matrices can be added, multiplied, and sometimes inverted. All methods for solving linear systems may be expressed as matrix manipulations using these operations. For example, solving the above system consists of computing an inverted matrix A − 1 {\displaystyle A^{-1}} such that A − 1 A = I , {\displaystyle A^{-1}A=I,} where I {\displaystyle I} is the identity matrix. Then, multiplying on the left both members of the above matrix equation by A − 1 , {\displaystyle A^{-1},} one gets the solution of the system of linear equations as[43] X = A − 1 B . {\displaystyle X=A^{-1}B.} Methods of solving systems of linear equations range from the introductory, like substitution[44] and elimination,[45] to more advanced techniques using matrices, such as Cramer's rule, the Gaussian elimination, and LU decomposition.[46] Some systems of equations are inconsistent, meaning that no solutions exist because the equations contradict each other.[47][f] Consistent systems have either one unique solution or an infinite number of solutions.[48][g] The study of vector spaces and linear maps form a large part of linear algebra. A vector space is an algebraic structure formed by a set with an addition that makes it an abelian group and a scalar multiplication that is compatible with addition (see vector space for details). A linear map is a function between vector spaces that is compatible with addition and scalar multiplication. In the case of finite-dimensional vector spaces, vectors and linear maps can be represented by matrices. It follows that the theories of matrices and finite-dimensional vector spaces are essentially the same. In particular, vector spaces provide a third way for expressing and manipulating systems of linear equations.[49] From this perspective, a matrix is a representation of a linear map: if one chooses a particular basis to describe the vectors being transformed, then the entries in the matrix give the results of applying the linear map to the basis vectors.[50] Linear equations with two variables can be interpreted geometrically as lines. The solution of a system of linear equations is where the lines intersect. Systems of equations can be interpreted as geometric figures. For systems with two variables, each equation represents a line in two-dimensional space. The point where the two lines intersect is the solution of the full system because this is the only point that solves both the first and the second equation. For inconsistent systems, the two lines run parallel, meaning that there is no solution since they never intersect. If two equations are not independent then they describe the same line, meaning that every solution of one equation is also a solution of the other equation. These relations make it possible to seek solutions graphically by plotting the equations and determining where they intersect.[51] The same principles also apply to systems of equations with more variables, with the difference being that the equations do not describe lines but higher dimensional figures. For instance, equations with three variables correspond to planes in three-dimensional space, and the points where all planes intersect solve the system of equations.[52] Main article: Abstract algebra Abstract algebra, also called modern algebra,[53] is the study of algebraic structures. An algebraic structure is a framework for understanding operations on mathematical objects, like the addition of numbers. While elementary algebra and linear algebra work within the confines of particular algebraic structures, abstract algebra takes a more general approach that compares how algebraic structures differ from each other and what types of algebraic structures there are, such as groups, rings, and fields.[54] The key difference between these types of algebraic structures lies in the number of operations they use and the laws they obey.[55] In mathematics education, abstract algebra refers to an advanced undergraduate course that mathematics majors take after completing courses in linear algebra.[56] Many algebraic structures rely on binary operations, which take two objects as their input and combine them into a single object as output, like addition and multiplication do. On a formal level, an algebraic structure is a set[h] of mathematical objects, called the underlying set, together with one or several operations.[i] Abstract algebra is primarily interested in binary operations,[j] which take any two objects from the underlying set as inputs and map them to another object from this set as output.[60] For example, the algebraic structure ⟨ N , + ⟩ {\displaystyle \langle \mathbb {N} ,+\rangle } has the natural numbers (N {\displaystyle \mathbb {N} }) as the underlying set and addition (+ {\displaystyle +}) as its binary operation.[58] The underlying set can contain mathematical objects other than numbers, and the operations are not restricted to regular arithmetic operations.[61] For instance, the underlying set of the symmetry group of a geometric object is made up of geometric transformations, such as rotations, under which the object remains unchanged. Its binary operation is function composition, which takes two transformations as input and has the transformation resulting from applying the first transformation followed by the second as its output.[62] Main article: Group theory Abstract algebra classifies algebraic structures based on the laws or axioms that its operations obey and the number of operations it uses. One of the most basic types is a group, which has one operation and requires that this operation is associative and has an identity element and inverse elements. An operation is associative if the order of several applications does not matter, i.e., if (a ∘ b) ∘ c {\displaystyle (a\circ b)\circ c} [k] is the same as a ∘ (b ∘ c) {\displaystyle a\circ (b\circ c)} for all elements. An operation has an identity element or a neutral element if one element e exists that does not change the value of any other element, i.e., if a ∘ e = e ∘ a = a {\displaystyle a\circ e=e\circ a=a} . An operation has inverse elements if for any element a {\displaystyle a} there exists a reciprocal element a − 1 {\displaystyle a^{-1}} that undoes a {\displaystyle a} . If an element operates on its inverse then the result is the neutral element e, expressed formally as a ∘ a − 1 = a − 1 ∘ a = e {\displaystyle a\circ a^{-1}=a^{-1}\circ a=e} . Every algebraic structure that fulfills these requirements is a group.[64] For example, ⟨ Z , + ⟩ {\displaystyle \langle \mathbb {Z} ,+\rangle } is a group formed by the set of integers together with the operation of addition. The neutral element is 0 and the inverse element of any number a {\displaystyle a} is − a {\displaystyle -a} .[65] The natural numbers with addition, by contrast, do not form a group since they contain only positive integers and therefore lack inverse elements.[66] Group theory examines the nature of groups, with basic theorems such as the fundamental theorem of finite abelian groups and the Feit–Thompson theorem.[67] The latter was a key early step in one of the most important mathematical achievements of the 20th century: the collaborative effort, taking up more than 10,000 journal pages and mostly published between 1960 and 2004, that culminated in a complete classification of finite simple groups.[68] Main articles: Ring theory and Field (mathematics) A ring is an algebraic structure with two operations that work similarly to the addition and multiplication of numbers and are named and generally denoted similarly. A ring is a commutative group under addition: the addition of the ring is associative, commutative, and has an identity element and inverse elements. The multiplication is associative and distributive with respect to addition; that is, a (b + c) = a b + a c {\displaystyle a(b+c)=ab+ac} and (b + c) a = b a + c a . {\displaystyle (b+c)a=ba+ca.} Moreover, multiplication is associative and has an identity element generally denoted as 1.[69][l] Multiplication needs not to be commutative; if it is commutative, one has a commutative ring.[71] The ring of integers (Z {\displaystyle \mathbb {Z} }) is one of the simplest commutative rings.[72] A field is a commutative ring such that 1 ≠ 0 {\displaystyle 1\neq 0} and each nonzero element has a multiplicative inverse.[73] The ring of integers does not form a field because it lacks multiplicative inverses. For example, the multiplicative inverse of 7 {\displaystyle 7} is 1 7 {\displaystyle {\tfrac {1}{7}}} , which is not an integer. The rational numbers, the real numbers, and the complex numbers each form a field with the operations of addition and multiplication.[74] Ring theory is the study of rings, exploring concepts such as subrings, quotient rings, polynomial rings, and ideals as well as theorems such as Hilbert's basis theorem.[75] Field theory is concerned with fields, examining field extensions, algebraic closures, and finite fields.[76] Galois theory explores the relation between field theory and group theory, relying on the fundamental theorem of Galois theory.[77] Diagram of relations between some algebraic structures. For instance, its top right section shows that a magma becomes a semigroup if its operation is associative. Besides groups, rings, and fields, there are many other algebraic structures studied by algebra. They include magmas, semigroups, monoids, abelian groups, commutative rings, modules, lattices, vector spaces, algebras over a field, and associative and non-associative algebras. They differ from each other regarding the types of objects they describe and the requirements that their operations fulfill. Many are related to each other in that a basic structure can be turned into a more specialized structure by adding constraints.[55] For example, a magma becomes a semigroup if its operation is associative.[78] Homomorphisms are tools to examine structural features by comparing two algebraic structures.[79] A homomorphism is a function from the underlying set of one algebraic structure to the underlying set of another algebraic structure that preserves certain structural characteristics. If the two algebraic structures use binary operations and have the form ⟨ A , ∘ ⟩ {\displaystyle \langle A,\circ \rangle } and ⟨ B , ⋆ ⟩ {\displaystyle \langle B,\star \rangle } then the function h : A → B {\displaystyle h:A\to B} is a homomorphism if it fulfills the following requirement: h (x ∘ y) = h (x) ⋆ h (y) {\displaystyle h(x\circ y)=h(x)\star h(y)} . The existence of a homomorphism reveals that the operation ⋆ {\displaystyle \star } in the second algebraic structure plays the same role as the operation ∘ {\displaystyle \circ } does in the first algebraic structure.[80] Isomorphisms are a special type of homomorphism that indicates a high degree of similarity between two algebraic structures. An isomorphism is a bijective homomorphism, meaning that it establishes a one-to-one relationship between the elements of the two algebraic structures. This implies that every element of the first algebraic structure is mapped to one unique element in the second structure without any unmapped elements in the second structure.[81] Subalgebras restrict their operations to a subset of the underlying set of the original algebraic structure. Another tool of comparison is the relation between an algebraic structure and its subalgebra.[82] The algebraic structure and its subalgebra use the same operations,[m] which follow the same axioms. The only difference is that the underlying set of the subalgebra is a subset of the underlying set of the algebraic structure.[n] All operations in the subalgebra are required to be closed in its underlying set, meaning that they only produce elements that belong to this set.[82] For example, the set of even integers together with addition is a subalgebra of the full set of integers together with addition. This is the case because the sum of two even numbers is again an even number. But the set of odd integers together with addition is not a subalgebra because it is not closed: adding two odd numbers produces an even number, which is not part of the chosen subset.[83] Universal algebra is the study of algebraic structures in general. As part of its general perspective, it is not concerned with the specific elements that make up the underlying sets and considers operations with more than two inputs, such as ternary operations. It provides a framework for investigating what structural features different algebraic structures have in common.[85][o] One of those structural features concerns the identities that are true in different algebraic structures. In this context, an identity is a universal equation or an equation that is true for all elements of the underlying set. For example, commutativity is a universal equation that states that a ∘ b {\displaystyle a\circ b} is identical to b ∘ a {\displaystyle b\circ a} for all elements.[87] A variety is a class of all algebraic structures that satisfy certain identities. For example, if two algebraic structures satisfy commutativity then they are both part of the corresponding variety.[88][p][q] Category theory examines how mathematical objects are related to each other using the concept of categories. A category is a collection of objects together with a collection of morphisms or "arrows" between those objects. These two collections must satisfy certain conditions. For example, morphisms can be joined, or composed: if there exists a morphism from object a {\displaystyle a} to object b {\displaystyle b} , and another morphism from object b {\displaystyle b} to object c {\displaystyle c} , then there must also exist one from object a {\displaystyle a} to object c {\displaystyle c} . Composition of morphisms is required to be associative, and there must be an "identity morphism" for every object.[92] Categories are widely used in contemporary mathematics since they provide a unifying framework to describe and analyze many fundamental mathematical concepts. For example, sets can be described with the category of sets, and any group can be regarded as the morphisms of a category with just one object.[93] Main articles: History of algebra and Timeline of algebra The Rhind Mathematical Papyrus from ancient Egypt, dated c. 1650 BCE, is one of the earliest documents discussing algebraic problems. The origin of algebra lies in attempts to solve mathematical problems involving arithmetic calculations and unknown quantities. These developments happened in the ancient period in Babylonia, Egypt, Greece, China, and India. One of the earliest documents on algebraic problems is the Rhind Mathematical Papyrus from ancient Egypt, which was written around 1650 BCE.[r] It discusses solutions to linear equations, as expressed in problems like "A quantity; its fourth is added to it. It becomes fifteen. What is the quantity?" Babylonian clay tablets from around the same time explain methods to solve linear and quadratic polynomial equations, such as the method of completing the square.[95] Many of these insights found their way to the ancient Greeks. Starting in the 6th century BCE, their main interest was geometry rather than algebra, but they employed algebraic methods to solve geometric problems. For example, they studied geometric figures while taking their lengths and areas as unknown quantities to be determined, as exemplified in Pythagoras' formulation of the difference of two squares method and later in Euclid's Elements.[96] In the 3rd century CE, Diophantus provided a detailed treatment of how to solve algebraic equations in a series of books called Arithmetica. He was the first to experiment with symbolic notation to express polynomials.[97] Diophantus's work influenced Arab development of algebra with many of his methods reflected in the concepts and techniques used in medieval Arabic algebra.[98] In ancient China, The Nine Chapters on the Mathematical Art, a book composed over the period spanning from the 10th century BCE to the 2nd century CE,[99] explored various techniques for solving algebraic equations, including the use of matrix-like constructs.[100] There is no unanimity of opinion as to whether these early developments are part of algebra or only precursors. They offered solutions to algebraic problems but did not conceive them in an abstract and general manner, focusing instead on specific cases and applications.[101] This changed with the Persian mathematician al-Khwarizmi,[s] who published his The Compendious Book on Calculation by Completion and Balancing in 825 CE. It presents the first detailed treatment of general methods that can be used to manipulate linear and quadratic equations by "reducing" and "balancing" both sides.[103] Other influential contributions to algebra came from the Arab mathematician Thābit ibn Qurra also in the 9th century and the Persian mathematician Omar Khayyam in the 11th and 12th centuries.[104] In India, Brahmagupta investigated how to solve quadratic equations and systems of equations with several variables in the 7th century CE. Among his innovations were the use of zero and negative numbers in algebraic equations.[105] The Indian mathematicians Mahāvīra in the 9th century and Bhāskara II in the 12th century further refined Brahmagupta's methods and concepts.[106] In 1247, the Chinese mathematician Qin Jiushao wrote the Mathematical Treatise in Nine Sections, which includes an algorithm for the numerical evaluation of polynomials, including polynomials of higher degrees.[107] François Viète (left) and René Descartes invented a symbolic notation to express equations in an abstract and concise manner. The Italian mathematician Fibonacci brought al-Khwarizmi's ideas and techniques to Europe in books including his Liber Abaci.[108] In 1545, the Italian polymath Gerolamo Cardano published his book Ars Magna, which covered many topics in algebra, discussed imaginary numbers, and was the first to present general methods for solving cubic and quartic equations.[109] In the 16th and 17th centuries, the French mathematicians François Viète and René Descartes introduced letters and symbols to denote variables and operations, making it possible to express equations in an concise and abstract manner. Their predecessors had relied on verbal descriptions of problems and solutions.[110] Some historians see this development as a key turning point in the history of algebra and consider what came before it as the prehistory of algebra because it lacked the abstract nature based on symbolic manipulation.[111] In the 17th and 18th centuries, many attempts were made to find general solutions to polynomials of degree five and higher. All of them failed.[37] At the end of the 18th century, the German mathematician Carl Friedrich Gauss proved the fundamental theorem of algebra, which describes the existence of zeros of polynomials of any degree without providing a general solution.[19] At the beginning of the 19th century, the Italian mathematician Paolo Ruffini and the Norwegian mathematician Niels Henrik Abel were able to show that no general solution exists for polynomials of degree five and higher.[37] In response to and shortly after their findings, the French mathematician Évariste Galois developed what came later to be known as Galois theory, which offered a more in-depth analysis of the solutions of polynomials while also laying the foundation of group theory.[20] Mathematicians soon realized the relevance of group theory to other fields and applied it to disciplines like geometry and number theory.[112] Garrett Birkhoff developed many of the foundational concepts of universal algebra. Starting in the mid-19th century, interest in algebra shifted from the study of polynomials associated with elementary algebra towards a more general inquiry into algebraic structures, marking the emergence of abstract algebra. This approach explored the axiomatic basis of arbitrary algebraic operations.[113] The invention of new algebraic systems based on different operations and elements accompanied this development, such as Boolean algebra, vector algebra, and matrix algebra.[114] Influential early developments in abstract algebra were made by the German mathematicians David Hilbert, Ernst Steinitz, and Emmy Noether as well as the Austrian mathematician Emil Artin. They researched different forms of algebraic structures and categorized them based on their underlying axioms into types, like groups, rings, and fields.[115] The idea of the even more general approach associated with universal algebra was conceived by the English mathematician Alfred North Whitehead in his 1898 book A Treatise on Universal Algebra. Starting in the 1930s, the American mathematician Garrett Birkhoff expanded these ideas and developed many of the foundational concepts of this field.[116] The invention of universal algebra led to the emergence of various new areas focused on the algebraization of mathematics—that is, the application of algebraic methods to other branches of mathematics. Topological algebra arose in the early 20th century, studying algebraic structures such as topological groups and Lie groups.[117] In the 1940s and 50s, homological algebra emerged, employing algebraic techniques to study homology.[118] Around the same time, category theory was developed and has since played a key role in the foundations of mathematics.[119] Other developments were the formulation of model theory and the study of free algebras.[120] See also: Applied mathematics The influence of algebra is wide-reaching, both within mathematics and in its applications to other fields.[121] The algebraization of mathematics is the process of applying algebraic methods and principles to other branches of mathematics, such as geometry, topology, number theory, and calculus. It happens by employing symbols in the form of variables to express mathematical insights on a more general level, allowing mathematicians to develop formal models describing how objects interact and relate to each other.[122] The algebraic equation x 2 + y 2 + z 2 = 1 {\displaystyle x^{2}+y^{2}+z^{2}=1} describes a sphere at the origin with a radius of 1. One application, found in geometry, is the use of algebraic statements to describe geometric figures. For example, the equation y = 3 x − 7 {\displaystyle y=3x-7} describes a line in two-dimensional space while the equation x 2 + y 2 + z 2 = 1 {\displaystyle x^{2}+y^{2}+z^{2}=1} corresponds to a sphere in three-dimensional space. Of special interest to algebraic geometry are algebraic varieties,[t] which are solutions to systems of polynomial equations that can be used to describe more complex geometric figures.[124] Algebraic reasoning can also solve geometric problems. For example, one can determine whether and where the line described by y = x + 1 {\displaystyle y=x+1} intersects with the circle described by x 2 + y 2 = 25 {\displaystyle x^{2}+y^{2}=25} by solving the system of equations made up of these two equations.[125] Topology studies the properties of geometric figures or topological spaces that are preserved under operations of continuous deformation. Algebraic topology relies on algebraic theories such as group theory to classify topological spaces. For example, homotopy groups classify topological spaces based on the existence of loops or holes in them.[126] Number theory is concerned with the properties of and relations between integers. Algebraic number theory applies algebraic methods and principles to this field of inquiry. Examples are the use of algebraic expressions to describe general laws, like Fermat's Last Theorem, and of algebraic structures to analyze the behavior of numbers, such as the ring of integers.[127] The related field of combinatorics uses algebraic techniques to solve problems related to counting, arrangement, and combination of discrete objects. An example in algebraic combinatorics is the application of group theory to analyze graphs and symmetries.[128] The insights of algebra are also relevant to calculus, which uses mathematical expressions to examine rates of change and accumulation. It relies on algebra, for instance, to understand how these expressions can be transformed and what role variables play in them.[129] Algebraic logic employs the methods of algebra to describe and analyze the structures and patterns that underlie logical reasoning,[130] exploring both the relevant mathematical structures themselves and their application to concrete problems of logic.[131] It includes the study of Boolean algebra to describe propositional logic[132] as well as the formulation and analysis of algebraic structures corresponding to more complex systems of logic.[133] The faces of a Rubik's cube can be rotated to change the arrangement of colored patches. The resulting permutations form a group called the Rubik's Cube group.[134] Algebraic methods are also commonly employed in other areas, like the natural sciences. For example, they are used to express scientific laws and solve equations in physics, chemistry, and biology.[135] Similar applications are found in fields like economics, geography, engineering (including electronics and robotics), and computer science to express relationships, solve problems, and model systems.[136] Linear algebra plays a central role in artificial intelligence and machine learning, for instance, by enabling the efficient processing and analysis of large datasets.[137] Various fields rely on algebraic structures investigated by abstract algebra. For example, physical sciences like crystallography and quantum mechanics make extensive use of group theory,[138] which is also employed to study puzzles such as Sudoku and Rubik's cubes,[139] and origami.[140] Both coding theory and cryptology rely on abstract algebra to solve problems associated with data transmission, like avoiding the effects of noise and ensuring data security.[141] See also: Mathematics education Balance scales are used in algebra education to help students understand how equations can be transformed to determine unknown values.[142] Algebra education mostly focuses on elementary algebra, which is one of the reasons why elementary algebra is also called school algebra. It is usually not introduced until secondary education since it requires mastery of the fundamentals of arithmetic while posing new cognitive challenges associated with abstract reasoning and generalization.[143] It aims to familiarize students with the formal side of mathematics by helping them understand mathematical symbolism, for example, how variables can be used to represent unknown quantities. An additional difficulty for students lies in the fact that, unlike arithmetic calculations, algebraic expressions are often difficult to solve directly. Instead, students need to learn how to transform them according to certain laws, often to determine an unknown quantity.[144] Some tools to introduce students to the abstract side of algebra rely on concrete models and visualizations of equations, including geometric analogies, manipulatives including sticks or cups, and "function machines" representing equations as flow diagrams. One method uses balance scales as a pictorial approach to help students grasp basic problems of algebra. The mass of some objects on the scale is unknown and represents variables. Solving an equation corresponds to adding and removing objects on both sides in such a way that the sides stay in balance until the only object remaining on one side is the object of unknown mass.[145] Word problems are another tool to show how algebra is applied to real-life situations. For example, students may be presented with a situation in which Naomi's brother has twice as many apples as Naomi. Given that both together have twelve apples, students are then asked to find an algebraic equation that describes this situation (2 x + x = 12 {\displaystyle 2x+x=12}) and to determine how many apples Naomi has (x = 4 {\displaystyle x=4}).[146] At the university level, mathematics students encounter advanced algebra topics from linear and abstract algebra. Initial undergraduate courses in linear algebra focus on matrices, vector spaces, and linear maps. Upon completing them, students are usually introduced to abstract algebra, where they learn about algebraic structures like groups, rings, and fields, as well as the relations between them. The curriculum typically also covers specific instances of algebraic structures, such as the systems of rational numbers, the real numbers, and the polynomials.[147] Commutative algebra – Branch of algebra that studies commutative rings Computer algebra – Scientific area at the interface between computer science and mathematics Exterior algebra – Algebra associated to any vector space F-algebra Geometric algebra – Algebraic structure designed for geometry Non-associative algebra – Algebra over a field where binary multiplication is not necessarily associative Outline of algebra – Overview of and topical guide to algebra Representation theory – Branch of mathematics that studies abstract algebraic structures Tensor – Algebraic object with geometric applications ^ When understood in the widest sense, an algebraic operation is a function from a Cartesian power of a set into that set, expressed formally as ω : A n → A {\displaystyle \omega :A^{n}\to A} . The addition of real numbers is an example of an algebraic operation: it takes two numbers as input and produces one number as output. It has the form + : R 2 → R {\displaystyle +:\mathbb {R} ^{2}\to \mathbb {R} } .[3] ^ Algebra is covered by division 512 in the Dewey Decimal Classification[5] and subclass QA 150-272.5 in the Library of Congress Classification.[6] It encompasses several areas in the Mathematics Subject Classification.[7] ^ The exact meaning of the term al-jabr in al-Khwarizmi's work is disputed. In some passages, it expresses that a quantity diminished by subtraction is restored to its original value, similar to how a bonesetter restores broken bones by bringing them into proper alignment.[17] ^ These changes were in part triggered by discoveries that solved many of the older problems of algebra. For example, the proof of the fundamental theorem of algebra demonstrated the existence of complex solutions of polynomials[19] and the introduction of Galois theory characterized the polynomials that have general solutions.[20] ^ Constants represent fixed numbers that do not change during the study of a specific problem.[24] ^ For example, the equations x 1 − 3 x 2 = 0 {\displaystyle x_{1}-3x_{2}=0} and x 1 − 3 x 2 = 7 {\displaystyle x_{1}-3x_{2}=7} contradict each other since no values of x 1 {\displaystyle x_{1}} and x 2 {\displaystyle x_{2}} exist that solve both equations at the same time.[47] ^ Whether a consistent system of equations has a unique solution depends on the number of variables and independent equations. Several equations are independent of each other if they do not provide the same information and cannot be derived from each other. A unique solution exists if the number of variables is the same as the number of independent equations. Underdetermined systems, by contrast, have more variables than independent equations and have an infinite number of solutions if they are consistent.[48] ^ A set is an unordered collection of distinct elements, such as numbers, vectors, or other sets. Set theory describes the laws and properties of sets.[57] ^ According to some definitions, algebraic structures include a distinguished element as an additional component, such as the identity element in the case of multiplication.[58] ^ Some of the algebraic structures studied by abstract algebra include unary operations in addition to binary operations. For example, normed vector spaces have a norm, which is a unary operation often used to associate a vector with its length.[59] ^ The symbols ∘ {\displaystyle \circ } and ⋆ {\displaystyle \star } are used in this article to represent any operation that may or may not resemble arithmetic operations.[63] ^ Some authors do not require the existence of multiplicative identity elements. A ring without multiplicative identity is sometimes called a rng.[70] ^ According to some definitions, it is also possible for a subalgebra to have fewer operations.[83] ^ This means that all the elements of the first set are also elements of the second set but the second set may contain elements not found in the first set.[84] ^ A slightly different approach understands universal algebra as the study of one type of algebraic structures known as universal algebras. Universal algebras are defined in a general manner to include most other algebraic structures. For example, groups and rings are special types of universal algebras.[86] ^ Not every type of algebraic structure forms a variety. For example, both groups and rings form varieties but fields do not.[89] ^ Besides identities, universal algebra is also interested in structural features associated with quasi-identities. A quasi-identity is an identity that only needs to be present under certain conditions (which take the form of a Horn clause[90]). It is a generalization of identity in the sense that every identity is a quasi-identity but not every quasi-identity is an identity. A quasivariety is a class of all algebraic structures that satisfy certain quasi-identities.[91] ^ The exact date is disputed and some historians suggest a later date around 1550 BCE.[94] ^ Some historians consider him the "father of algebra" while others reserve this title for Diophantus.[102] ^ Algebraic varieties studied in geometry differ from the more general varieties studied in universal algebra.[123] ^ Merzlyakov & Shirshov 2020, Lead sectionGilbert & Nicholson 2004, p. 4 ^ Fiche & Hebuterne 2013, p. 326Merzlyakov & Shirshov 2020, § The Subject Matter of Algebra, Its Principal Branches and Its Connection with Other Branches of Mathematics.Gilbert & Nicholson 2004, p. 4 ^ Baranovich 2023, Lead section ^ Pratt 2022, Lead section, § 1. Elementary Algebra, § 2. Abstract Algebra, § 3. 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ISBN 978-0-471-75684-2. Retrieved January 16, 2024. Young, Cynthia Y. (2023). Precalculus. John Wiley & Sons. ISBN 978-1-119-86940-5. Retrieved January 18, 2024. zbMATH Open (2024). "Classification". zbMATH Open. Mathematical Reviews and zbMATH Open. Archived from the original on July 19, 2020. Retrieved March 17, 2024. Zwillinger, Daniel (2002). CRC Standard Mathematical Tables and Formulae. CRC Press. ISBN 978-1-4200-3534-6. Retrieved January 27, 2024. Wikiquote has quotations related to Algebra. Look up algebra in Wiktionary, the free dictionary. Wikibooks has a book on the topic of: Algebra Wikisource has the text of the 1911 Encyclopædia Britannica article "Algebra". Retrieved from " For the modern history of algebra, see Abstract algebra § History. Algebra can essentially be considered as doing computations similar to those of arithmetic but with non-numerical mathematical objects. However, until the 19th century, algebra consisted essentially of the theory of equations. For example, the fundamental theorem of algebra belongs to the theory of equations and is not, nowadays, considered as belonging to algebra (in fact, every proof must use the completeness of the real numbers, which is not an algebraic property). This article describes the history of the theory of equations, referred to in this article as "algebra", from the origins to the emergence of algebra as a separate area of mathematics. The word "algebra" is derived from the Arabic word الجبر al-jabr, and this comes from the treatise written in the year 830 by the medieval Persian mathematician, Al-Khwārizmī, whose Arabic title, Kitāb al-muḫtaṣar fī ḥisāb al-ğabr wa-l-muqābala, can be translat as The Compendious Book on Calculation by Completion and Balancing. The treatise provided for the systematic solution of linear and quadratic equations. According to one history, "[i]t is not certain just what the terms al-jabr and muqabalah mean, but the usual interpretation is similar to that implied in the previous translation. The word 'al-jabr' presumably meant something like 'restoration' or 'completion' and seems to refer to the transposition of subtracted terms to the other side of an equation; the word 'muqabalah' is said to refer to 'reduction' or 'balancing'—that is, the cancellation of like terms on opposite sides of the equation. Arabic influence in Spain long after the time of al-Khwarizmi is found in Don Quixote, where the word 'algebrista' is used for a bone-setter, that is, a 'restorer'."[1] The term is used by al-Khwarizmi to describe the operations that he introduced, "reduction" and "balancing", referring to the transposition of subtracted terms to the other side of an equation, that is, the cancellation of like terms on opposite sides of the equation.[2] See also: Timeline of algebra Algebra did not always make use of the symbolism that is now ubiquitous in mathematics; instead, it went through three distinct stages. The stages in the development of symbolic algebra are approximately as follows:[3] Rhetorical algebra, in which equations are written in full sentences. For example, the rhetorical form of x + 1 = 2 {\displaystyle x+1=2} is "The thing plus one equals two" or possibly "The thing plus 1 equals 2". Rhetorical algebra was first developed by the ancient Babylonians and remained dominant up to the 16th century. Syncopated algebra, in which some symbolism is used, but which does not contain all of the characteristics of symbolic algebra. For instance, there may be a restriction that subtraction may be used only once within one side of an equation, which is not the case with symbolic algebra. Syncopated algebraic expression first appeared in Diophantus' Arithmetica (3rd century AD), followed by Brahmagupta's Brahma Sphuta Siddhanta (7th century). Symbolic algebra, in which full symbolism is used. Early steps toward this can be seen in the work of several Islamic mathematicians such as Ibn al-Banna (13th–14th centuries) and al-Qalasadi (15th century), although fully symbolic algebra was developed by François Viète (16th century). Later, René Descartes (17th century) introduced the modern notation (for example, the use of x—see below) and showed that the problems occurring in geometry can be expressed and solved in terms of algebra (Cartesian geometry). As important as the use or lack of symbolism in algebra was the degree of the equations that were addressed. Quadratic equations played an important role in early algebra; and throughout most of history, until the early modern period, all quadratic equations were classified as belonging to one of three categories. x 2 + p x = q {\displaystyle x^{2}+px=q} x 2 = p x + q {\displaystyle x^{2}=px+q} x 2 + q = p x {\displaystyle x^{2}+q=px} where p {\displaystyle p} and q {\displaystyle q} are positive. This trichotomy comes about because quadratic equations of the form x 2 + p x + q = 0 {\displaystyle x^{2}+px+q=0} , with p {\displaystyle p} and q {\displaystyle q} positive, have no positive roots.[4] In between the rhetorical and syncopated stages of symbolic algebra, a geometric constructive algebra was developed by classical Greek and Vedic Indian mathematicians in which algebraic equations were solved through geometry. For instance, an equation of the form x 2 = A {\displaystyle x^{2}=A} was solved by finding the side of a square of area A . {\displaystyle A.} In addition to the three stages of expressing algebraic ideas, some authors recognized four conceptual stages in the development of algebra that occurred alongside the changes in expression. These four stages were as follows:[5] Geometric stage, where the concepts of algebra are largely geometric. This dates back to the Babylonians and continued with the Greeks, and was later revived by Omar Khayyām. Static equation-solving stage, where the objective is to find numbers satisfying certain relationships. The move away from the geometric stage dates back to Diophantus and Brahmagupta, but algebra did not decisively move to the static equation-solving stage until Al-Khwarizmi introduced generalized algorithmic processes for solving algebraic problems. Dynamic function stage, where motion is an underlying idea. The idea of a function began emerging with Sharaf al-Dīn al-Tūsī, but algebra did not decisively move to the dynamic function stage until Gottfried Leibniz. Abstract stage, where mathematical structure plays a central role. Abstract algebra is largely a product of the 19th and 20th centuries. See also: Babylonian mathematics The Plimpton 322 tablet The origins of algebra can be traced to the ancient Babylonians,[6] who developed a positional number system that greatly aided them in solving their rhetorical algebraic equations. The Babylonians were not interested in exact solutions, but rather approximations, and so they would commonly use linear interpolation to approximate intermediate values.[7] One of the most famous tablets is the Plimpton 322 tablet, created around 1900–1600 BC, which gives a table of Pythagorean triples and represents some of the most advanced mathematics prior to Greek mathematics.[8] Babylonian algebra was much more advanced than the Egyptian algebra of the time; whereas the Egyptians were mainly concerned with linear equations the Babylonians were more concerned with quadratic and cubic equations.[7] The Babylonians had developed flexible algebraic operations with which they were able to add equals to equals and multiply both sides of an equation by like quantities so as to eliminate fractions and factors.[7] They were familiar with many simple forms of factoring,[7] three-term quadratic equations with positive roots,[9] and many cubic equations,[10] although it is not known if they were able to reduce the general cubic equation.[10] A portion of the Rhind Papyrus See also: Egyptian mathematics Ancient Egyptian algebra dealt mainly with linear equations while the Babylonians found these equations too elementary, and developed mathematics to a higher level than the Egyptians.[7] The Rhind Papyrus, also known as the Ahmes Papyrus, is an ancient Egyptian papyrus written c. 1650 BC by Ahmes, who transcribed it from an earlier work that he dated to between 2000 and 1800 BC.[11] It is the most extensive ancient Egyptian mathematical document known to historians.[12] The Rhind Papyrus contains problems where linear equations of the form x + a x = b {\displaystyle x+ax=b} and x + a x + b x = c {\displaystyle x+ax+bx=c} are solved, where a , b , {\displaystyle a,b,} and c {\displaystyle c} are known and x , {\displaystyle x,} which is referred to as "aha" or heap, is the unknown. [13] The solutions were possibly, but not likely, arrived at by using the "method of false position", or regula falsi, where first a specific value is substituted into the left hand side of the equation, then the required arithmetic calculations are done, thirdly the result is compared to the right hand side of the equation, and finally the correct answer is found through the use of proportions. In some of the problems the author "checks" his solution, thereby writing one of the earliest known simple proofs.[13] See also: Greek mathematics One of the oldest surviving fragments of Euclid's Elements, found at Oxyrhynchus and dated to circa 100 AD (P. Oxy. 29). The diagram accompanies Book II, Proposition 5. [14] It is sometimes alleged that the Greeks had no algebra, but this is disputed.[15] By the time of Plato, Greek mathematics had undergone a drastic change. The Greeks created a geometric algebra where terms were represented by sides of geometric objects,[16] usually lines, that had letters associated with them,[17] and with this new form of algebra they were able to find solutions to equations by using a process that they invented, known as "the application of areas".[16] "The application of areas" is only a part of geometric algebra and it is thoroughly covered in Euclid's Elements. An example of geometric algebra would be solving the linear equation a x = b c . {\displaystyle ax=bc.} The ancient Greeks would solve this equation by looking at it as an equality of areas rather than as an equality between the ratios a : b {\displaystyle a:b} and c : x . {\displaystyle c:x.} The Greeks would construct a rectangle with sides of length b {\displaystyle b} and c , {\displaystyle c,} then extend a side of the rectangle to length a , {\displaystyle a,} and finally they would complete the extended rectangle so as to find the side of the rectangle that is the solution.[16] Iamblichus in Introductio arithmatica says that Thymaridas (c. 400 BC – c. 350 BC) worked with simultaneous linear equations.[18] In particular, he created the then famous rule that was known as the "bloom of Thymaridas" or as the "flower of Thymaridas", which states that: If the sum of n {\displaystyle n} quantities be given, and also the sum of every pair containing a particular quantity, then this particular quantity is equal to 1 / (n − 2) {\displaystyle 1/(n-2)} of the difference between the sums of these pairs and the first given sum.[19] A proof from Euclid's Elements that, given a line segment, an equilateral triangle exists that includes the segment as one of its sides or using modern notation, the solution of the following system of n {\displaystyle n} linear equations in n {\displaystyle n} unknowns,[18] x + x 1 + x 2 + ⋯ + x n − 1 = s {\displaystyle x+x_{1}+x_{2}+\cdots +x_{n-1}=s} x + x 1 = m 1 {\displaystyle x+x_{1}=m_{1}} x + x 2 = m 2 {\displaystyle x+x_{2}=m_{2}} ⋮ {\displaystyle \vdots } x + x n − 1 = m n − 1 {\displaystyle x+x_{n-1}=m_{n-1}} is, x = (m 1 + m 2 + . . . + m n − 1) − s n − 2 = (∑ i = 1 n − 1 m i) − s n − 2 . {\displaystyle x={\cfrac {(m_{1}+m_{2}+...+m_{n-1})-s}{n-2}}={\cfrac {(\sum _{i=1}^{n-1}m_{i})-s}{n-2}}.} Iamblichus goes on to describe how some systems of linear equations that are not in this form can be placed into this form.[18] Hellenistic mathematician Euclid details geometrical algebra. Euclid (Greek: Εὐκλείδης) was a Greek mathematician who flourished in Alexandria, Egypt, almost certainly during the reign of Ptolemy I (323–283 BC).[20][21] Neither the year nor place of his birth[20] have been established, nor the circumstances of his death. Euclid is regarded as the "father of geometry". His Elements is the most successful textbook in the history of mathematics.[20] Although he is one of the most famous mathematicians in history there are no new discoveries attributed to him; rather he is remembered for his great explanatory skills.[22] The Elements is not, as is sometimes thought, a collection of all Greek mathematical knowledge to its date; rather, it is an elementary introduction to it.[23] The geometric work of the Greeks, typified in Euclid's Elements, provided the framework for generalizing formulae beyond the solution of particular problems into more general systems of stating and solving equations. Book II of the Elements contains fourteen propositions, which in Euclid's time were extremely significant for doing geometric algebra. These propositions and their results are the geometric equivalents of our modern symbolic algebra and trigonometry.[15] Today, using modern symbolic algebra, we let symbols represent known and unknown magnitudes (i.e. numbers) and then apply algebraic operations on them, while in Euclid's time magnitudes were viewed as line segments and then results were deduced using the axioms or theorems of geometry.[15] Many basic laws of addition and multiplication are included or proved geometrically in the Elements. For instance, proposition 1 of Book II states: If there be two straight lines, and one of them be cut into any number of segments whatever, the rectangle contained by the two straight lines is equal to the rectangles contained by the uncut straight line and each of the segments. But this is nothing more than the geometric version of the (left) distributive law, a (b + c + d) = a b + a c + a d {\displaystyle a(b+c+d)=ab+ac+ad} ; and in Books V and VII of the Elements the commutative and associative laws for multiplication are demonstrated.[15] Many basic equations were also proved geometrically. For instance, proposition 5 in Book II proves that a 2 − b 2 = (a + b) (a − b) , {\displaystyle a^{2}-b^{2}=(a+b)(a-b),} [24] and proposition 4 in Book II proves that (a + b) 2 = a 2 + 2 a b + b 2 . {\displaystyle (a+b)^{2}=a^{2}+2ab+b^{2}.} [15] Furthermore, there are also geometric solutions given to many equations. For instance, proposition 6 of Book II gives the solution to the quadratic equation a x + x 2 = b 2 , {\displaystyle ax+x^{2}=b^{2},} and proposition 11 of Book II gives a solution to a x + x 2 = a 2 . {\displaystyle ax+x^{2}=a^{2}.} [25] Data is a work written by Euclid for use at the schools of Alexandria and it was meant to be used as a companion volume to the first six books of the Elements. The book contains some fifteen definitions and ninety-five statements, of which there are about two dozen statements that serve as algebraic rules or formulas.[26] Some of these statements are geometric equivalents to solutions of quadratic equations.[26] For instance, Data contains the solutions to the equations d x 2 − a d x + b 2 c = 0 {\displaystyle dx^{2}-adx+b^{2}c=0} and the familiar Babylonian equation x y = a 2 , x ± y = b . {\displaystyle xy=a^{2},x\pm y=b.} [26] A conic section is a curve that results from the intersection of a cone with a plane. There are three primary types of conic sections: ellipses (including circles), parabolas, and hyperbolas. The conic sections are reputed to have been discovered by Menaechmus[27] (c. 380 BC – c. 320 BC) and since dealing with conic sections is equivalent to dealing with their respective equations, they played geometric roles equivalent to cubic equations and other higher order equations. Menaechmus knew that in a parabola, the equation y 2 = l x {\displaystyle y^{2}=lx} holds, where l {\displaystyle l} is a constant called the latus rectum, although he was not aware of the fact that any equation in two unknowns determines a curve.[28] He apparently derived these properties of conic sections and others as well. Using this information it was now possible to find a solution to the problem of the duplication of the cube by solving for the points at which two parabolas intersect, a solution equivalent to solving a cubic equation.[28] We are informed by Eutocius that the method he used to solve the cubic equation was due to Dionysodorus (250 BC – 190 BC). Dionysodorus solved the cubic by means of the intersection of a rectangular hyperbola and a parabola. This was related to a problem in Archimedes' On the Sphere and Cylinder. Conic sections would be studied and used for thousands of years by Greek, and later Islamic and European, mathematicians. In particular Apollonius of Perga's famous Conics deals with conic sections, among other topics. See also: Chinese mathematics Chinese mathematics dates to at least 300 BC with the Zhoubi Suanjing, generally considered to be one of the oldest Chinese mathematical documents.[29] Nine Chapters on the Mathematical Art Chiu-chang suan-shu or The Nine Chapters on the Mathematical Art, written around 250 BC, is one of the most influential of all Chinese math books and it is composed of some 246 problems. Chapter eight deals with solving determinate and indeterminate simultaneous linear equations using positive and negative numbers, with one problem dealing with solving four equations in five unknowns.[29] Ts'e-yuan hai-ching, or Sea-Mirror of the Circle Measurements, is a collection of some 170 problems written by Li Zhi (or Li Ye) (1192 – 1279 AD). He used fan fa, or Horner's method, to solve equations of degree as high as six, although he did not describe his method of solving equations.[30] Shu-shu chiu-chang, or Mathematical Treatise in Nine Sections, was written by the wealthy governor and minister Ch'in Chiu-shao (c. 1202 – c. 1261). With the introduction of a method for solving simultaneous congruences, now called the Chinese remainder theorem, it marks the high point in Chinese indeterminate analysis[clarification needed].[30] Yang Hui (Pascal's) triangle, as depicted by the ancient Chinese using rod numerals The earliest known magic squares appeared in China.[31] In Nine Chapters the author solves a system of simultaneous linear equations by placing the coefficients and constant terms of the linear equations into a magic square (i.e. a matrix) and performing column reducing operations on the magic square.[31] The earliest known magic squares of order greater than three are attributed to Yang Hui (fl. c. 1261 – 1275), who worked with magic squares of order as high as ten.[32] Ssy-yüan yü-chien《四元玉鑒》, or Precious Mirror of the Four Elements, was written by Chu Shih-chieh in 1303 and it marks the peak in the development of Chinese algebra. The four elements, called heaven, earth, man and matter, represented the four unknown quantities in his algebraic equations. The Ssy-yüan yü-chien deals with simultaneous equations and with equations of degrees as high as fourteen. The author uses the method of fan fa, today called Horner's method, to solve these equations.[33] The Precious Mirror opens with a diagram of the arithmetic triangle (Pascal's triangle) using a round zero symbol, but Chu Shih-chieh denies credit for it. A similar triangle appears in Yang Hui's work, but without the zero symbol.[34] There are many summation equations given without proof in the Precious mirror. A few of the summations are:[34] 1 2 + 2 2 + 3 2 + ⋯ + n 2 = n (n + 1) (2 n + 1) 3 ! {\displaystyle 1^{2}+2^{2}+3^{2}+\cdots +n^{2}={n(n+1)(2n+1) \over 3!}} 1 + 8 + 30 + 80 + ⋯ + n 2 (n + 1) (n + 2) 3 ! = n (n + 1) (n + 2) (n + 3) (4 n + 1) 5 ! {\displaystyle 1+8+30+80+\cdots +{n^{2}(n+1)(n+2) \over 3!}={n(n+1)(n+2)(n+3)(4n+1) \over 5!}} See also: Diophantine equation and Arithmetica Cover of the 1621 edition of Diophantus' Arithmetica, translated into Latin by Claude Gaspard Bachet de Méziriac Diophantus was a Hellenistic mathematician who lived c. 250 AD, but the uncertainty of this date is so great that it may be off by more than a century. He is known for having written Arithmetica, a treatise that was originally thirteen books but of which only the first six have survived.[35] Arithmetica is the earliest extant work present that solve arithmetic problems by algebra. Diophantus however did not invent the method of algebra, which existed before him.[36] Algebra was practiced and diffused orally by practitioners, with Diophantus picking up techniques to solve problems in arithmetic.[37] In modern algebra a polynomial is a linear combination of variable x that is built of exponentiation, scalar multiplication, addition, and subtraction. The algebra of Diophantus, similar to medieval arabic algebra is an aggregation of objects of different types with no operations present[38] For example, in Diophantus a polynomial "6 4′ inverse Powers, 25 Powers lacking 9 units", which in modern notation is 6 1 4 x − 1 + 25 x 2 − 9 {\displaystyle 6{\tfrac {1}{4}}x^{-1}+25x^{2}-9} is a collection of 6 1 4 {\displaystyle 6{\tfrac {1}{4}}} object of one kind with 25 object of second kind which lack 9 objects of third kind with no operation present.[39] Similar to medieval Arabic algebra Diophantus uses three stages to solve a problem by Algebra: 1) An unknown is named and an equation is set up 2) An equation is simplified to a standard form(al-jabr and al-muqābala in arabic) 3) Simplified equation is solved[40] Diophantus does not give a classification of equations in six types like Al-Khwarizmi in extant parts of Arithmetica. He does say that he would give solution to three terms equations later, so this part of the work is possibly just lost[37] In Arithmetica, Diophantus is the first to use symbols for unknown numbers as well as abbreviations for powers of numbers, relationships, and operations;[41] thus he used what is now known as syncopated algebra. The main difference between Diophantine syncopated algebra and modern algebraic notation is that the former lacked special symbols for operations, relations, and exponentials.[42] So, for example, what we would write as x 3 − 2 x 2 + 10 x − 1 = 5 , {\displaystyle x^{3}-2x^{2}+10x-1=5,} which can be rewritten as (x 3 1 + x 10) − (x 2 2 + x 0 1) = x 0 5 , {\displaystyle \left({x^{3}}1+{x}10\right)-\left({x^{2}}2+{x^{0}}1\right)={x^{0}}5,} would be written in Diophantus's syncopated notation as K υ α ¯ ζ ι ¯ ⋔ Δ υ β ¯ M α ¯ ἴ σ M ε ¯ {\displaystyle \mathrm {K} ^{\upsilon }{\overline {\alpha }}\;\zeta {\overline {\iota }}\;\,\pitchfork \;\,\Delta ^{\upsilon }{\overline {\beta }}\;\mathrm {M} {\overline {\alpha }}\,\;{\sigma \;\,\mathrm {M} {\overline {\varepsilon }}} where the symbols represent the following:[43][44] Symbol What it represents α ¯ {\displaystyle {\overline {\alpha }}} 1 β ¯ {\displaystyle {\overline {\beta }}} 2 ε ¯ {\displaystyle {\overline {\varepsilon }}} 5 ι ¯ {\displaystyle {\overline {\iota }}} 10 ἴσ "equals" (short for ἴσος) ⋔ {\displaystyle \pitchfork } represents the subtraction of everything that follows ⋔ {\displaystyle \pitchfork } up to ἴσ M {\displaystyle \mathrm {M} } the zeroth power (i.e. a constant term) ζ {\displaystyle \zeta } the unknown quantity (because a number x {\displaystyle x} raised to the first power is just x , {\displaystyle x,} this may be thought of as "the first power") Δ υ {\displaystyle \Delta ^{\upsilon }} the second power, from Greek δύναμις, meaning strength or power K υ {\displaystyle \mathrm {K} ^{\upsilon }} the third power, from Greek κύβος, meaning a cube Δ υ Δ {\displaystyle \Delta ^{\upsilon }\Delta } the fourth power Δ K υ {\displaystyle \Delta \mathrm {K} ^{\upsilon }} the fifth power K υ K {\displaystyle \mathrm {K} ^{\upsilon }\mathrm {K} } the sixth power Unlike in modern notation, the coefficients come after the variables and that addition is represented by the juxtaposition of terms. A literal symbol-for-symbol translation of Diophantus's syncopated equation into a modern symbolic equation would be the following:[43] x 3 1 x 10 − x 2 2 x 0 1 = x 0 5 {\displaystyle {x^{3}}1{x}10-{x^{2}}2{x^{0}}1={x^{0}}5} where to clarify, if the modern parentheses and plus are used then the above equation can be rewritten as:[43] (x 3 1 + x 10) − (x 2 2 + x 0 1) = x 0 5 {\displaystyle \left({x^{3}}1+{x}10\right)-\left({x^{2}}2+{x^{0}}1\right)={x^{0}}5} However the distinction between "rhetorical algebra", "syncopated algebra" and "symbolic algebra" is considered outdated by Jeffrey Oaks and Jean Christianidis. The problems were solved on dust-board using some notation, while in books solution were written in "rhetorical style".[45] Arithmetica also makes use of the identities:[46] (a 2 + b 2) (c 2 + d 2) {\displaystyle (a^{2}+b^{2})(c^{2}+d^{2})} = (a c + d b) 2 + (b c − a d) 2 {\displaystyle =(ac+db)^{2}+(bc-ad)^{2}} = (a d + b c) 2 + (a c − b d) 2 {\displaystyle =(ad+bc)^{2}+(ac-bd)^{2}} See also: Indian mathematics The Indian mathematicians were active in studying about number systems. The earliest known Indian mathematical documents are dated to around the middle of the first millennium BC (around the 6th century BC).[47] The recurring themes in Indian mathematics are, among others, determinate and indeterminate linear and quadratic equations, simple mensuration, and Pythagorean triples.[48] Aryabhata (476–550) was an Indian mathematician who authored Aryabhatiya. In it he gave the rules,[49] 1 2 + 2 2 + ⋯ + n 2 = n (n + 1) (2 n + 1) 6 {\displaystyle 1^{2}+2^{2}+\cdots +n^{2}={n(n+1)(2n+1) \over 6}} and 1 3 + 2 3 + ⋯ + n 3 = (1 + 2 + ⋯ + n) 2 {\displaystyle 1^{3}+2^{3}+\cdots +n^{3}=(1+2+\cdots +n)^{2}} Brahmagupta (fl. 628) was an Indian mathematician who authored Brahma Sphuta Siddhanta. In his work Brahmagupta solves the general quadratic equation for both positive and negative roots.[50] In indeterminate analysis Brahmagupta gives the Pythagorean triads m , 1 2 (m 2 n − n) , {\displaystyle m,{\frac {1}{2}}\left({m^{2} \over n}-n\right),} 1 2 (m 2 n + n) , {\displaystyle {\frac {1}{2}}\left({m^{2} \over n}+n\right),} but this is a modified form of an old Babylonian rule that Brahmagupta may have been familiar with.[51] He was the first to give a general solution to the linear Diophantine equation a x + b y = c , {\displaystyle ax+by=c,} where a , b , {\displaystyle a,b,} and c {\displaystyle c} are integers. Unlike Diophantus who only gave one solution to an indeterminate equation, Brahmagupta gave all integer solutions; but that Brahmagupta used some of the same examples as Diophantus has led some historians to consider the possibility of a Greek influence on Brahmagupta's work, or at least a common Babylonian source.[52] Like the algebra of Diophantus, the algebra of Brahmagupta was syncopated. Addition was indicated by placing the numbers side by side, subtraction by placing a dot over the subtrahend, and division by placing the divisor below the dividend, similar to our notation but without the bar. Multiplication, evolution, and unknown quantities were represented by abbreviations of appropriate terms.[52] The extent of Greek influence on this syncopation, if any, is not known and it is possible that both Greek and Indian syncopation may be derived from a common Babylonian source. [52] Bhāskara II (1114 – c. 1185) was the leading mathematician of the 12th century. In Algebra, he gave the general solution of Pell's equation.[52] He is the author of Lilavati and Vija-Ganita, which contain problems dealing with determinate and indeterminate linear and quadratic equations, and Pythagorean triples[48] and he fails to distinguish between exact and approximate statements.[53] Many of the problems in Lilavati and Vija-Ganita are derived from other Hindu sources, and so Bhaskara is at his best in dealing with indeterminate analysis.[53] Bhaskara uses the initial symbols of the names for colors as the symbols of unknown variables. So, for example, what we would write today as (− x − 1) + (2 x − 8) = x − 9 {\displaystyle (-x-1)+(2x-8)=x-9} Bhaskara would have written as . _ . ya 1 ru 1 . ya 2 ru 8 . Sum ya 1 ru 9 where ya indicates the first syllable of the word for black, and ru is taken from the word species. The dots over the numbers indicate subtraction. A page from The Compendious Book on Calculation by Completion and Balancing See also: Islamic mathematics The first century of the Islamic Arab Empire saw almost no scientific or mathematical achievements since the Arabs, with their newly conquered empire, had not yet gained any intellectual drive and research in other parts of the world had faded. In the second half of the 8th century, Islam had a cultural awakening, and research in mathematics and the sciences increased.[54] The Muslim Abbasid caliph al-Mamun (809–833) is said to have had a dream where Aristotle appeared to him, and as a consequence al-Mamun ordered that Arabic translation be made of as many Greek works as possible, including Ptolemy's Almagest and Euclid's Elements. Greek works would be given to the Muslims by the Byzantine Empire in exchange for treaties, as the two empires held an uneasy peace.[54] Many of these Greek works were translated by Thabit ibn Qurra (826–901), who translated books written by Euclid, Archimedes, Apollonius, Ptolemy, and Eutocius.[55] Arabic mathematicians established algebra as an independent discipline, and gave it the name "algebra" (al-jabr). They were the first to teach algebra in an elementary form and for its own sake.[56] There are three theories about the origins of Arabic Algebra. The first emphasizes Hindu influence, the second emphasizes Mesopotamian or Persian-Syriac influence and the third emphasizes Greek influence. Many scholars believe that it is the result of a combination of all three sources.[57] Throughout their time in power, the Arabs used a fully rhetorical algebra, where often even the numbers were spelled out in words. The Arabs would eventually replace spelled out numbers (e.g. twenty-two) with Arabic numerals (e.g. 22), but the Arabs did not adopt or develop a syncopated or symbolic algebra[55] until the work of Ibn al-Banna, who developed a symbolic algebra in the 13th century, followed by Abū al-Hasan ibn Alī al-Qalasādī in the 15th century. See also: The Compendious Book on Calculation by Completion and Balancing Left: The original Arabic print manuscript of the Book of Algebra by Al-Khwarizmi. Right: A page from The Algebra of Al-Khwarizmi by Fredrick Rosen, in English. The Muslim[58] Persian mathematician Muhammad ibn Mūsā al-Khwārizmī, described as the father[59][60][61] or founder[62][63] of algebra, was a faculty member of the "House of Wisdom" (Bait al-Hikma) in Baghdad, which was established by Al-Mamun. Al-Khwarizmi, who died around 850 AD, wrote more than half a dozen mathematical and astronomical works.[54] One of al-Khwarizmi's most famous books is entitled Al-jabr wa'l muqabalah or The Compendious Book on Calculation by Completion and Balancing, and it gives an exhaustive account of solving polynomials up to the second degree.[64] The book also introduced the fundamental concept of "reduction" and "balancing", referring to the transposition of subtracted terms to the other side of an equation, that is, the cancellation of like terms on opposite sides of the equation. This is the operation which Al-Khwarizmi originally described as al-jabr.[65] The name "algebra" comes from the "al-jabr" in the title of his book. R. Rashed and Angela Armstrong write: "Al-Khwarizmi's text can be seen to be distinct not only from the Babylonian tablets, but also from Diophantus' Arithmetica. It no longer concerns a series of problems to be resolved, but an exposition which starts with primitive terms in which the combinations must give all possible prototypes for equations, which henceforward explicitly constitute the true object of study. On the other hand, the idea of an equation for its own sake appears from the beginning and, one could say, in a generic manner, insofar as it does not simply emerge in the course of solving a problem, but is specifically called on to define an infinite class of problems."[66] Al-Jabr is divided into six chapters, each of which deals with a different type of formula. The first chapter of Al-Jabr deals with equations whose squares equal its roots (a x 2 = b x) , {\displaystyle \left(ax^{2}=bx\right),} the second chapter deals with squares equal to number (a x 2 = c) , {\displaystyle \left(ax^{2}=c\right),} the third chapter deals with roots equal to a number (b x = c) , {\displaystyle \left(bx=c\right),} the fourth chapter deals with squares and roots equal a number (a x 2 + b x = c) , {\displaystyle \left(ax^{2}+bx=c\right),} the fifth chapter deals with squares and number equal roots (a x 2 + c = b x) , {\displaystyle \left(ax^{2}+c=bx\right),} and the sixth and final chapter deals with roots and number equal to squares (b x + c = a x 2) . {\displaystyle \left(bx+c=ax^{2}\right).} [67] Pages from a 14th-century Arabic copy of the book, showing geometric solutions to two quadratic equations In Al-Jabr, al-Khwarizmi uses geometric proofs,[17] he does not recognize the root x = 0 , {\displaystyle x=0,} [67] and he only deals with positive roots.[68] He also recognizes that the discriminant must be positive and described the method of completing the square, though he does not justify the procedure.[69] The Greek influence is shown by Al-Jabr's geometric foundations[57][70] and by one problem taken from Heron.[71] He makes use of lettered diagrams but all of the coefficients in all of his equations are specific numbers since he had no way of expressing with parameters what he could express geometrically; although generality of method is intended.[17] Al-Khwarizmi most likely did not know of Diophantus's Arithmetica,[72] which became known to the Arabs sometime before the 10th century.[73] And even though al-Khwarizmi most likely knew of Brahmagupta's work, Al-Jabr is fully rhetorical with the numbers even being spelled out in words.[72] So, for example, what we would write as x 2 + 10 x = 39 {\displaystyle x^{2}+10x=39} Diophantus would have written as[74] Δ Υ α ¯ ς ι ¯ {\displaystyle \Delta ^{\Upsilon }{\overline {\alpha }}\varsigma {\overline {\iota }}\,\;} ἴ σ M λ θ ¯ {\displaystyle \sigma \;\,\mathrm {M} \lambda {\overline {\theta }}} And al-Khwarizmi would have written as[74] One square and ten roots of the same amount to thirty-nine dirhems; that is to say, what must be the square which, when increased by ten of its own roots, amounts to thirty-nine? 'Abd al-Hamīd ibn Turk authored a manuscript entitled Logical Necessities in Mixed Equations, which is very similar to al-Khwarizmi's Al-Jabr and was published at around the same time as, or even possibly earlier than, Al-Jabr.[73] The manuscript gives exactly the same geometric demonstration as is found in Al-Jabr, and in one case the same example as found in Al-Jabr, and even goes beyond Al-Jabr by giving a geometric proof that if the discriminant is negative then the quadratic equation has no solution.[73] The similarity between these two works has led some historians to conclude that Arabic algebra may have been well developed by the time of al-Khwarizmi and 'Abd al-Hamid.[73] Arabic mathematicians treated irrational numbers as algebraic objects.[75] The Egyptian mathematician Abū Kāmil Shujā ibn Aslam (c. 850–930) was the first to accept irrational numbers in the form of a square root or fourth root as solutions to quadratic equations or as coefficients in an equation.[76] He was also the first to solve three non-linear simultaneous equations with three unknown variables.[77] Al-Karaji (953–1029), also known as Al-Karkhi, was the successor of Abū al-Wafā' al-Būzjānī (940–998) and he discovered the first numerical solution to equations of the form a x 2 n + b x n = c . {\displaystyle ax^{2n}+bx^{n}=c.} [78] Al-Karaji only considered positive roots.[78] He is also regarded as the first person to free algebra from geometrical operations and replace them with the type of arithmetic operations which are at the core of algebra today. His work on algebra and polynomials gave the rules for arithmetic operations to manipulate polynomials. The historian of mathematics F. Woepcke, in Extrait du Fakhri, traité d'Algèbre par Abou Bekr Mohammed Ben Alhacan Alkarkhi (Paris, 1853), praised Al-Karaji for being "the first who introduced the theory of algebraic calculus". Stemming from this, Al-Karaji investigated binomial coefficients and Pascal's triangle.[79] Omar Khayyám To solve the third-degree equation x 3 + a 2 x = b {\displaystyle x^{3}+a^{2}x=b} Khayyám constructed the parabola x 2 = a y , {\displaystyle x^{2}=ay,} a circle with diameter b / a 2 , {\displaystyle b/a^{2},} and a vertical line through the intersection point. The solution is given by the length of the horizontal line segment from the origin to the intersection of the vertical line and the x {\displaystyle x} -axis. Omar Khayyám (c. 1050 – 1123) wrote a book on Algebra that went beyond Al-Jabr to include equations of the third degree.[80] Omar Khayyám provided both arithmetic and geometric solutions for quadratic equations, but he only gave geometric solutions for general cubic equations since he mistakenly believed that arithmetic solutions were impossible.[80] His method of solving cubic equations by using intersecting conics had been used by Menaechmus, Archimedes, and Ibn al-Haytham (Alhazen), but Omar Khayyám generalized the method to cover all cubic equations with positive roots.[80] He only considered positive roots and he did not go past the third degree.[80] He also saw a strong relationship between geometry and algebra.[80] In the 12th century, Sharaf al-Dīn al-Tūsī (1135–1213) wrote the Al-Mu'adalat (Treatise on Equations), which dealt with eight types of cubic equations with positive solutions and five types of cubic equations which may not have positive solutions. He used what would later be known as the "Ruffini-Horner method" to numerically approximate the root of a cubic equation. He also developed the concepts of the maxima and minima of curves in order to solve cubic equations which may not have positive solutions.[81] He understood the importance of the discriminant of the cubic equation and used an early version of Cardano's formula[82] to find algebraic solutions to certain types of cubic equations. Some scholars, such as Roshdi Rashed, argue that Sharaf al-Din discovered the derivative of cubic polynomials and realized its significance, while other scholars connect his solution to the ideas of Euclid and Archimedes.[83] Sharaf al-Din also developed the concept of a function.[84] In his analysis of the equation x 3 + d = b x 2 {\displaystyle x^{3}+d=bx^{2}} for example, he begins by changing the equation's form to x 2 (b − x) = d {\displaystyle x^{2}(b-x)=d} . He then states that the question of whether the equation has a solution depends on whether or not the "function" on the left side reaches the value d {\displaystyle d} . To determine this, he finds a maximum value for the function. He proves that the maximum value occurs when x = 2 b 3 {\displaystyle \textstyle x={\frac {2b}{3}}} , which gives the functional value 4 b 3 27 {\displaystyle \textstyle {\frac {4b^{3}}{27}}} . Sharaf al-Din then states that if this value is less than d {\displaystyle d} , there are no positive solutions; if it is equal to d {\displaystyle d} , then there is one solution at x = 2 b 3 {\displaystyle \textstyle x={\frac {2b}{3}}} ; and if it is greater than d {\displaystyle d} , then there are two solutions, one between 0 {\displaystyle 0} and 2 b 3 {\displaystyle \textstyle {\frac {2b}{3}}} and one between 2 b 3 {\displaystyle \textstyle {\frac {2b}{3}}} and b {\displaystyle b} . [85] In the early 15th century, Jamshīd al-Kāshī developed an early form of Newton's method to numerically solve the equation x P − N = 0 {\displaystyle x^{P}-N=0} to find roots of N {\displaystyle N} .[86] Al-Kāshī also developed decimal fractions and claimed to have discovered it himself. However, J. Lennart Berggrenn notes that he was mistaken, as decimal fractions were first used five centuries before him by the Baghdadi mathematician Abu'l-Hasan al-Uqlidisi as early as the 10th century.[77] Al-Hassār, a mathematician from Morocco specializing in Islamic inheritance jurisprudence during the 12th century, developed the modern symbolic mathematical notation for fractions, where the numerator and denominator are separated by a horizontal bar. This same fractional notation appeared soon after in the work of Fibonacci in the 13th century.[87][failed verification] Abū al-Ḥasan ibn ʿAlī al-Qalaṣādī (1412–1486) was the last major medieval Arab algebraist, who made the first attempt at creating an algebraic notation since Ibn al-Banna two centuries earlier, who was himself the first to make such an attempt since Diophantus and Brahmagupta in ancient times.[88] The syncopated notations of his predecessors, however, lacked symbols for mathematical operations.[42] Al-Qalasadi "took the first steps toward the introduction of algebraic symbolism by using letters in place of numbers"[88] and by "using short Arabic words, or just their initial letters, as mathematical symbols."[88] Just as the death of Hypatia signals the close of the Library of Alexandria as a mathematical center, so does the death of Boethius signal the end of mathematics in the Western Roman Empire. Although there was some work being done at Athens, it came to a close when in 529 the Byzantine emperor Justinian closed the pagan philosophical schools. The year 529 is now taken to be the beginning of the medieval period. Scholars fled the West towards the more hospitable East, particularly towards Persia, where they found haven under King Chosroes and established what might be termed an "Athenian Academy in Exile".[89] Under a treaty with Justinian, Chosroes would eventually return the scholars to the Eastern Empire. During the Dark Ages, European mathematics was at its nadir with mathematical research consisting mainly of commentaries on ancient treatises; and most of this research was centered in the Byzantine Empire. The end of the medieval period is set as the fall of Constantinople to the Turks in 1453. The 12th century saw a flood of translations from Arabic into Latin and by the 13th century, European mathematics was beginning to rival the mathematics of other lands. In the 13th century, the solution of a cubic equation by Fibonacci is representative of the beginning of a revival in European algebra. As the Islamic world was declining after the 15th century, the European world was ascending. And it is here that algebra was further developed. This section is missing information about most important results in algebra from the 15th century, and are completely ignored. Please expand the section to include this information. Further details may exist on the talk page. (January 2017) Modern notation for arithmetic operations was introduced between the end of the 15th century and the beginning of the 16th century by Johannes Widmann and Michael Stifel. At the end of 16th century, François Viète introduced symbols, now called variables, for representing indeterminate or unknown numbers. This created a new algebra consisting of computing with symbolic expressions as if they were numbers. Another key event in the further development of algebra was the general algebraic solution of the cubic and quartic equations, developed in the mid-16th century. The idea of a determinant was developed by Japanese mathematician Kowa Seki in the 17th century, followed by Gottfried Leibniz ten years later, for the purpose of solving systems of simultaneous linear equations using matrices. Gabriel Cramer also did some work on matrices and determinants in the 18th century. By tradition, the first unknown variable in an algebraic problem is nowadays represented by the symbol x {\displaystyle {\mathit {x}}} and if there is a second or a third unknown, then these are labeled y {\displaystyle {\mathit {y}}} and z {\displaystyle {\mathit {z}}} respectively. Algebraic x {\displaystyle x} is conventionally printed in italic type to distinguish it from the sign of multiplication. Mathematical historians[90] generally agree that the use of x {\displaystyle x} in algebra was introduced by René Descartes and was first published in his treatise La Géométrie (1637).[91][92] In that work, he used letters from the beginning of the alphabet (a , b , c , …) {\displaystyle (a,b,c,\ldots)} for known quantities, and letters from the end of the alphabet (z , y , x , …) {\displaystyle (z,y,x,\ldots)} for unknowns.[93] It has been suggested that he later settled on x {\displaystyle x} (in place of z {\displaystyle z}) for the first unknown because of its relatively greater abundance in the French and Latin typographical fonts of the time.[94] Three alternative theories of the origin of algebraic x were suggested in the 19th century: (1) a symbol used by German algebraists and thought to be derived from a cursive letter r , {\displaystyle r,} mistaken for x {\displaystyle x} ; (2) the numeral 1 with oblique strikethrough; and (3) an Arabic/Spanish source (see below). But the Swiss-American historian of mathematics Florian Cajori examined these and found all three lacking in concrete evidence; Cajori credited Descartes as the originator, and described his x , y , {\displaystyle x,y,} and z {\displaystyle z} as "free from tradition[,] and their choice purely arbitrary."[97] Nevertheless, the Hispano-Arabic hypothesis continues to have a presence in popular culture today.[98] It is the claim that algebraic x {\displaystyle x} is the abbreviation of a supposed loanword from Arabic in Old Spanish. The theory originated in 1884 with the German orientalist Paul de Lagarde, shortly after he published his edition of a 1505 Spanish/Arabic bilingual glossary[99] in which Spanish cosa ("thing") was paired with its Arabic equivalent, شيء (shay?) transcribed as xei. (The "sh" sound in Old Spanish was routinely spelled x . {\displaystyle x.}) Evidently Lagarde was aware that Arab mathematicians, in the "rhetorical" stage of algebra's development, often used that word to represent the unknown quantity. He surmised that "nothing could be more natural" ("Nichts war also natürlicher...") than for the initial of the Arabic word—romanized as the Old Spanish x {\displaystyle x} —to be adopted for use in algebra.[100] A later reader reinterpreted Lagarde's conjecture as having "proven" the point.[101] Lagarde was unaware that early Spanish mathematicians used, not a transcription of the Arabic word, but rather its translation in their own language, "cosa".[102] There is no instance of xei or similar forms in several compiled historical vocabularies of Spanish.[103][104] Although the mathematical notion of function was implicit in trigonometric and logarithmic tables, which existed in his day, Gottfried Leibniz was the first, in 1692 and 1694, to employ it explicitly, to denote any of several geometric concepts derived from a curve, such as abscissa, ordinate, tangent, chord, and the perpendicular.[105] In the 18th century, "function" lost these geometrical associations. Leibniz realized that the coefficients of a system of linear equations could be arranged into an array, now called a matrix, which can be manipulated to find the solution of the system, if any. This method was later called Gaussian elimination. Leibniz also discovered Boolean algebra and symbolic logic, also relevant to algebra. The ability to do algebra is a skill cultivated in mathematics education. As explained by Andrew Warwick, Cambridge University students in the early 19th century practiced "mixed mathematics",[106] doing exercises based on physical variables such as space, time, and weight. Over time the association of variables with physical quantities faded away as mathematical technique grew. Eventually mathematics was concerned completely with abstract polynomials, complex numbers, hypercomplex numbers and other concepts. Application to physical situations was then called applied mathematics or mathematical physics, and the field of mathematics expanded to include abstract algebra. For instance, the issue of constructible numbers showed some mathematical limitations, and the field of Galois theory was developed. The title of "the father of algebra" is frequently credited to the Persian mathematician Al-Khwarizmi,[107][108][109] supported by historians of mathematics, such as Carl Benjamin Boyer,[107] Solomon Gandz and Bartel Leendert van der Waerden.[110] However, the point is debatable and the title is sometimes credited to the Hellenistic mathematician Diophantus.[107][111] Those who support Diophantus point to the algebra found in Al-Jabr being more elementary than the algebra found in Arithmetica, and Arithmetica being syncopated while Al-Jabr is fully rhetorical.[107] However, the mathematics historian Kurt Vogel argues against Diophantus holding this title,[112] as his mathematics was not much more algebraic than that of the ancient Babylonians.[113] Those who support Al-Khwarizmi point to the fact that he gave an exhaustive explanation for the algebraic solution of quadratic equations with positive roots,[114] and was the first to teach algebra in an elementary form and for its own sake, whereas Diophantus was primarily concerned with the theory of numbers.[56] Al-Khwarizmi also introduced the fundamental concept of "reduction" and "balancing" (which he originally used the term al-jabr to refer to), referring to the transposition of subtracted terms to the other side of an equation, that is, the cancellation of like terms on opposite sides of the equation.[65] Other supporters of Al-Khwarizmi point to his algebra no longer being concerned "with a series of problems to be resolved, but an exposition which starts with primitive terms in which the combinations must give all possible prototypes for equations, which henceforward explicitly constitute the true object of study." They also point to his treatment of an equation for its own sake and "in a generic manner, insofar as it does not simply emerge in the course of solving a problem, but is specifically called on to define an infinite class of problems".[66] Victor J. Katz regards Al-Jabr as the first true algebra text that is still extant.[115] According to Jeffrey Oaks and Jean Christianidis neither Diophantus nor Al-Khwarizmi should be called "father of algebra".[116][117] Pre-modern algebra was developed and used by merchants and surveyors as part of what Jens Høyrup called "subscientific" tradition. Diophantus used this method of algebra in his book, in particular for indeterminate problems, while Al-Khwarizmi wrote one of the first books in Arabic about this method.[37] Mathematics portal Al-Mansur – 2nd Abbasid caliph (r. 754–775) Timeline of algebra – Notable events in the history of algebra ^ Boyer (1991:229) ^ Jeffrey A. Oaks; Haitham M. Alkhateeb (2007). "Simplifying equations in Arabic algebra". Historia Mathematica. 34: 45–61. doi:10.1016/j.hm.2006.02.006. ISSN 0315-0860. ^ (Boyer 1991, "Revival and Decline of Greek Mathematics" p. 180) "It has been said that three stages of in the historical development of algebra can be recognized: (1) the rhetorical or early stage, in which everything is written out fully in words; (2) a syncopated or intermediate state, in which some abbreviations are adopted; and (3) a symbolic or final stage. Such an arbitrary division of the development of algebra into three stages is, of course, a facile oversimplification; but it can serve effectively as a first approximation to what has happened"" ^ (Boyer 1991, "Mesopotamia" p. 32) "Until modern times there was no thought of solving a quadratic equation of the form x 2 + p x + q = 0 {\displaystyle x^{2}+px+q=0} , where p {\displaystyle p} and q {\displaystyle q} are positive, for the equation has no positive root. Consequently, quadratic equations in ancient and Medieval times—and even in the early modern period—were classified under three types: (1) x 2 + p x = q {\displaystyle x^{2}+px=q} (2) x 2 = p x + q {\displaystyle x^{2}=px+q} (3) x 2 + q = p x {\displaystyle x^{2}+q=px} " ^ Katz, Victor J.; Barton, Bill (October 2007), "Stages in the History of Algebra with Implications for Teaching", Educational Studies in Mathematics, 66 (2): 185–201, doi:10.1007/s10649-006-9023-7, S2CID 120363574 ^ Struik, Dirk J. (1987). A Concise History of Mathematics. New York: Dover Publications. ISBN 978-0-486-60255-4. ^ a b c d e (Boyer 1991, "Mesopotamia" p. 30) "Babylonian mathematicians did not hesitate to interpolate by proportional parts to approximate intermediate values. Linear interpolation seems to have been a commonplace procedure in ancient Mesopotamia, and the positional notation lent itself conveniently to the rule of three. [...] a table essential in Babylonian algebra; this subject reached a considerably higher level in Mesopotamia than in Egypt. Many problem texts from the Old Babylonian period show that the solution of the complete three-term quadratic equation afforded the Babylonians no serious difficulty, for flexible algebraic operations had been developed. They could transpose terms in an equations by adding equals to equals, and they could multiply both sides by like quantities to remove fractions or to eliminate factors. By adding 4 a b {\displaystyle 4ab} to (a − b) 2 {\displaystyle (a-b)^{2}} they could obtain (a + b) 2 {\displaystyle (a+b)^{2}} for they were familiar with many simple forms of factoring. [...]Egyptian algebra had been much concerned with linear equations, but the Babylonians evidently found these too elementary for much attention. [...] In another problem in an Old Babylonian text we find two simultaneous linear equations in two unknown quantities, called respectively the "first silver ring" and the "second silver ring"." ^ Joyce, David E. (1995). "Plimpton 322". The clay tablet with the catalog number 322 in the G. A. Plimpton Collection at Columbia University may be the most well known mathematical tablet, certainly the most photographed one, but it deserves even greater renown. It was scribed in the Old Babylonian period between -1900 and -1600 and shows the most advanced mathematics before the development of Greek mathematics. ^ (Boyer 1991, "Mesopotamia" p. 31) "The solution of a three-term quadratic equation seems to have exceeded by far the algebraic capabilities of the Egyptians, but Neugebauer in 1930 disclosed that such equations had been handled effectively by the Babylonians in some of the oldest problem texts." ^ a b (Boyer 1991, "Mesopotamia" p. 33) "There is no record in Egypt of the solution of a cubic equations, but among the Babylonians there are many instances of this. [...] Whether or not the Babylonians were able to reduce the general four-term cubic, ax3 + bx2 + cx = d, to their normal form is not known." ^ (Boyer 1991, "Egypt" p. 11) "It had been bought in 1959 in a Nile resort town by a Scottish antiquary, Henry Rhind; hence, it often is known as the Rhind Papyrus or, less frequently, as the Ahmes Papyrus in honor of the scribe by whose hand it had been copied in about 1650 BC. The scribe tells us that the material is derived from a prototype from the Middle Kingdom of about 2000 to 1800 BCE." ^ (Boyer 1991, "Egypt" p. 19) "Much of our information about Egyptian mathematics has been derived from the Rhind or Ahmes Papyrus, the most extensive mathematical document from ancient Egypt; but there are other sources as well." ^ a b (Boyer 1991, "Egypt" pp. 15–16) "The Egyptian problems so far described are best classified as arithmetic, but there are others that fall into a class to which the term algebraic is appropriately applied. These do not concern specific concrete objects such as bread and beer, nor do they call for operations on known numbers. Instead they require the equivalent of solutions of linear equations of the form x + a x = b {\displaystyle x+ax=b} or x + a x + b x = c {\displaystyle x+ax+bx=c} , where a and b and c are known and x is unknown. The unknown is referred to as "aha", or heap. [...] The solution given by Ahmes is not that of modern textbooks, but one proposed characteristic of a procedure now known as the "method of false position", or the "rule of false". A specific false value has been proposed by 1920s scholars and the operations indicated on the left hand side of the equality sign are performed on this assumed number. Recent scholarship shows that scribes had not guessed in these situations. Exact rational number answers written in Egyptian fraction series had confused the 1920s scholars. The attested result shows that Ahmes "checked" result by showing that 16 + 1/2 + 1/8 exactly added to a seventh of this (which is 2 + 1/4 + 1/8), does obtain 19. Here we see another significant step in the development of mathematics, for the check is a simple instance of a proof." ^ Bill Casselman. "One of the Oldest Extant Diagrams from Euclid". University of British Columbia. Retrieved 2008-09-26. ^ a b c d e (Boyer 1991, "Euclid of Alexandria" p.109) "Book II of the Elements is a short one, containing only fourteen propositions, not one of which plays any role in modern textbooks; yet in Euclid's day this book was of great significance. This sharp discrepancy between ancient and modern views is easily explained—today we have symbolic algebra and trigonometry that have replaced the geometric equivalents from Greece. For instance, Proposition 1 of Book II states that "If there be two straight lines, and one of them be cut into any number of segments whatever, the rectangle contained by the two straight lines is equal to the rectangles contained by the uncut straight line and each of the segments." This theorem, which asserts (Fig. 7.5) that AD (AP + PR + RB) = AD·AP + AD·PR + AD·RB, is nothing more than a geometric statement of one of the fundamental laws of arithmetic known today as the distributive law: a (b + c + d) = a b + a c + a d . {\displaystyle a(b+c+d)=ab+ac+ad.} In later books of the Elements (V and VII) we find demonstrations of the commutative and associative laws for multiplication. Whereas in our time magnitudes are represented by letters that are understood to be numbers (either known or unknown) on which we operate with algorithmic rules of algebra, in Euclid's day magnitudes were pictured as line segments satisfying the axions and theorems of geometry. It is sometimes asserted that the Greeks had no algebra, but this is patently false. They had Book II of the Elements, which is geometric algebra and served much the same purpose as does our symbolic algebra. There can be little doubt that modern algebra greatly facilitates the manipulation of relationships among magnitudes. But it is undoubtedly also true that a Greek geometer versed in the fourteen theorems of Euclid's "algebra" was far more adept in applying these theorems to practical mensuration than is an experienced geometer of today. Ancient geometric "algebra" was not an ideal tool, but it was far from ineffective. Euclid's statement (Proposition 4), "If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments, is a verbose way of saying that (a + b) 2 = a 2 + 2 a b + b 2 {\displaystyle (a+b)^{2}=a^{2}+2ab+b^{2}} ," ^ a b c (Boyer 1991, "The Heroic Age" pp. 77–78) "Whether deduction came into mathematics in the sixth century BCE or the fourth and whether incommensurability was discovered before or after 400 BCE, there can be no doubt that Greek mathematics had undergone drastic changes by the time of Plato. [...] A "geometric algebra" had to take the place of the older "arithmetic algebra", and in this new algebra there could be no adding of lines to areas or of areas to volumes. From now on there had to be strict homogeneity of terms in equations, and the Mesopotamian normal form, x y = A , x ± y = b , {\displaystyle xy=A,x\pm y=b,} = b, were to be interpreted geometrically. [...] In this way the Greeks built up the solution of quadratic equations by their process known as "the application of areas", a portion of geometric algebra that is fully covered by Euclid's Elements. [...] The linear equation a x = b c , {\displaystyle ax=bc,} , for example, was looked upon as an equality of the areas a x {\displaystyle ax} and b c , {\displaystyle bc,} rather than as a proportion—an equality between the two ratios a : b {\displaystyle a:b} and c : x . {\displaystyle c:x.} . Consequently, in constructing the fourth proportion x {\displaystyle x} in this case, it was usual to construct a rectangle OCDB with the sides b = OB and c = OC (Fig 5.9) and then along OC to lay off OA = a. One completes the rectangle OCDB and draws the diagonal OE cutting CD in P. It is now clear that CP is the desired line x , {\displaystyle x,} for rectangle OARS is equal in area to rectangle OCDB" ^ a b c (Boyer 1991, "Europe in the Middle Ages" p. 258) "In the arithmetical theorems in Euclid's Elements VII–IX, numbers had been represented by line segments to which letters had been attached, and the geometric proofs in al-Khwarizmi's Algebra made use of lettered diagrams; but all coefficients in the equations used in the Algebra are specific numbers, whether represented by numerals or written out in words. The idea of generality is implied in al-Khwarizmi's exposition, but he had no scheme for expressing algebraically the general propositions that are so readily available in geometry." ^ a b c (Heath 1981a, "The ('Bloom') of Thymaridas" pp. 94–96) Thymaridas of Paros, an ancient Pythagorean already mentioned (p. 69), was the author of a rule for solving a certain set of n {\displaystyle n} simultaneous simple equations connecting n {\displaystyle n} unknown quantities. The rule was evidently well known, for it was called by the special name [...] the 'flower' or 'bloom' of Thymaridas. [...] The rule is very obscurely worded, but it states in effect that, if we have the following n {\displaystyle n} equations connecting n {\displaystyle n} unknown quantities x , x 1 , x 2 , ... , x n − 1 , {\displaystyle x,x_{1},x_{2},\ldots ,x_{n-1},} namely [...] Iamblichus, our informant on this subject, goes on to show that other types of equations can be reduced to this, so that they rule does not 'leave us in the lurch' in those cases either." ^ (Flegg 1983, "Unknown Numbers" p. 205) "Thymaridas (fourth century) is said to have had this rule for solving a particular set of n {\displaystyle n} linear equations in n {\displaystyle n} unknowns:If the sum of n {\displaystyle n} quantities be given, and also the sum of every pair containing a particular quantity, then this particular quantity is equal to 1 / (n − 2) {\displaystyle 1/(n-2)} of the difference between the sums of these pairs and the first given sum." ^ a b c (Boyer 1991, "Euclid of Alexandria" p. 100) "but by 306 BCE control of the Egyptian portion of the empire was firmly in the hands of Ptolemy I, and this enlightened ruler was able to turn his attention to constructive efforts. Among his early acts was the establishment at Alexandria of a school or institute, known as the Museum, second to none in its day. As teachers at the school he called a band of leading scholars, among whom was the author of the most fabulously successful mathematics textbook ever written—the Elements (Stoichia) of Euclid. Considering the fame of the author and of his best seller, remarkably little is known of Euclid's life. So obscure was his life that no birthplace is associated with his name." ^ (Boyer 1991, "Euclid of Alexandria" p. 101) "The tale related above in connection with a request of Alexander the Great for an easy introduction to geometry is repeated in the case of Ptolemy, who Euclid is reported to have assured that "there is no royal road to geometry"." ^ (Boyer 1991, "Euclid of Alexandria" p. 104) "Some of the faculty probably excelled in research, others were better fitted to be administrators, and still some others were noted for teaching ability. It would appear, from the reports we have, that Euclid very definitely fitted into the last category. There is no new discovery attributed to him, but he was noted for expository skills." ^ (Boyer 1991, "Euclid of Alexandria" p. 104) "The Elements was not, as is sometimes thought, a compendium of all geometric knowledge; it was instead an introductory textbook covering all elementary mathematics." ^ (Boyer 1991, "Euclid of Alexandria" p. 110) "The same holds true for Elements II.5, which contains what we should regard as an impractical circumlocution for a 2 − b 2 = (a + b) (a − b) {\displaystyle a^{2}-b^{2}=(a+b)(a-b)} " ^ (Boyer 1991, "Euclid of Alexandria" p. 111) "In an exactly analogous manner the quadratic equation a x + x 2 = b 2 {\displaystyle ax+x^{2}=b^{2}} is solved through the use of II.6: If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole (with the added straight line) and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line. [...] with II.11 being an important special case of II.6. Here Euclid solves the equation a x + x 2 = a 2 {\displaystyle ax+x^{2}=a^{2}} " ^ a b c (Boyer 1991, "Euclid of Alexandria" p. 103) "Euclid's Data, a work that has come down to us through both Greek and the Arabic. It seems to have been composed for use at the schools of Alexandria, serving as a companion volume to the first six books of the Elements in much the same way that a manual of tables supplements a textbook. [...] It opens with fifteen definitions concerning magnitudes and loci. The body of the text comprises ninety-five statements concerning the implications of conditions and magnitudes that may be given in a problem. [...] There are about two dozen similar statements serving as algebraic rules or formulas. [...] Some of the statements are geometric equivalents of the solution of quadratic equations. For example[...] Eliminating y {\displaystyle y} we have (a − x) d x = b 2 c {\displaystyle (a-x)dx=b^{2}c} or d x 2 − a d x + b 2 c = 0 , {\displaystyle dx^{2}-adx+b^{2}c=0,} from which x = a 2 ± (a 2) 2 − b 2 (c d) . {\displaystyle x={\frac {a}{2}}\pm {\sqrt {\left({\frac {a}{2}}\right)^{2}-b^{2}\left({\frac {c}{d}}\right)}}.} The geometric solution given by Euclid is equivalent to this, except that the negative sign before the radical is used. Statements 84 and 85 in the Data are geometric replacements of the familiar Babylonian algebraic solutions of the systems x y = a 2 , x ± y = b , {\displaystyle xy=a^{2},x\pm y=b,} which again are the equivalents of solutions of simultaneous equations." ^ (Boyer 1991, "The Euclidean Synthesis" p. 103) "Eutocius and Proclus both attribute the discovery of the conic sections to Menaechmus, who lived in Athens in the late fourth century BC. Proclus, quoting Eratosthenes, refers to "the conic section triads of Menaechmus." Since this quotation comes just after a discussion of "the section of a right-angled cone" and "the section of an acute-angled cone", it is inferred that the conic sections were produced by cutting a cone with a plane perpendicular to one of its elements. Then if the vertex angle of the cone is acute, the resulting section (calledoxytome) is an ellipse. If the angle is right, the section (orthotome) is a parabola, and if the angle is obtuse, the section (amblytome) is a hyperbola (see Fig. 5.7)." ^ a b (Boyer 1991, "The age of Plato and Aristotle" p. 94–95) "If OP = y and OD = x are coordinates of point P, we have y 2 = R {\displaystyle y^{2}=R}).OV, or, on substituting equals, y2 = R'D.OV = AR'.BC/AB.DO.BC/AB = AR'.BC2/AB2.xInasmuch as segments AR', BC, and AB are the same for all points P on the curve EQDPG, we can write the equation of the curve, a "section of a right-angled cone", as y 2 = l x , {\displaystyle y^{2}=lx,} where l {\displaystyle l} is a constant, later to be known as the latus rectum of the curve. [...] Menaechmus apparently derived these properties of the conic sections and others as well. Since this material has a string resemblance to the use of coordinates, as illustrated above, it has sometimes been maintains that Menaechmus had analytic geometry. Such a judgment is warranted only in part, for certainly Menaechmus was unaware that any equation in two unknown quantities determines a curve. In fact, the general concept of an equation in unknown quantities was alien to Greek thought. [...] He had hit upon the conics in a successful search for curves with the properties appropriate to the duplication of the cube. In terms of modern notation the solution is easily achieved. By shifting the curring plane (Gig. 6.2), we can find a parabola with any latus rectum. If, then, we wish to duplicate a cube of edge a , {\displaystyle a,} we locate on a right-angled cone two parabolas, one with latus rectum a {\displaystyle a} and another with latus rectum 2 a . {\displaystyle 2a.} [...] It is probable that Menaechmus knew that the duplication could be achieved also by the use of a rectangular hyperbola and a parabola." ^ a b (Boyer 1991, "China and India" pp. 195–197) "estimates concerning the Chou Pei Suan Ching, generally considered to be the oldest of the mathematical classics, differ by almost a thousand years. [...] A date of about 300 B.C. would appear reasonable, thus placing it in close competition with another treatise, the Chiu-chang suan-shu, composed about 250 B.C., that is, shortly before the Han dynasty (202 B.C.). [...] Almost as old at the Chou Pei, and perhaps the most influential of all Chinese mathematical books, was the Chui-chang suan-shu, or Nine Chapters on the Mathematical Art. This book includes 246 problems on surveying, agriculture, partnerships, engineering, taxation, calculation, the solution of equations, and the properties of right triangles. [...] Chapter eight of the Nine chapters is significant for its solution of problems of simultaneous linear equations, using both positive and negative numbers. The last problem in the chapter involves four equations in five unknowns, and the topic of indeterminate equations was to remain a favorite among Oriental peoples." ^ a b (Boyer 1991, "China and India" p. 204) "Li Chih (or Li Yeh, 1192–1279), a mathematician of Peking who was offered a government post by Khublai Khan in 1206, but politely found an excuse to decline it. His Ts'e-yuan hai-ching (Sea-Mirror of the Circle Measurements) includes 170 problems dealing with[...]some of the problems leading to equations of fourth degree. Although he did not describe his method of solution of equations, including some of sixth degree, it appears that it was not very different form that used by Chu Shih-chieh and Horner. Others who used the Horner method were Ch'in Chiu-shao (c. 1202 – c. 1261) and Yang Hui (fl. c. 1261 – 1275). The former was an unprincipled governor and minister who acquired immense wealth within a hundred days of assuming office. His Shu-shu chiu-chang (Mathematical Treatise in Nine Sections) marks the high point of Chinese indeterminate analysis, with the invention of routines for solving simultaneous congruences." ^ a b (Boyer 1991, "China and India" p. 197) "The Chinese were especially fond of patters; hence, it is not surprising that the first record (of ancient but unknown origin) of a magic square appeared there. [...] The concern for such patterns left the author of the Nine Chapters to solve the system of simultaneous linear equations [...] by performing column operations on the matrix [...] to reduce it to [...] The second form represented the equations 36 z = 99 , 5 y + z = 24 , {\displaystyle 36z=99,5y+z=24,} = 24, and 3 x + 2 y + z = 39 {\displaystyle 3x+2y+z=39} from which the values of z , y , {\displaystyle z,y,} and x {\displaystyle x} are successively found with ease." ^ (Boyer 1991, "China and India" pp. 204–205) "The same "Horner" device was used by Yang Hui, about whose life almost nothing is known and who work has survived only in part. Among his contributions that are extant are the earliest Chinese magic squares of order greater than three, including two each of orders four through eight and one each of orders nine and ten." ^ (Boyer 1991, "China and India" p. 203) "The last and greatest of the Sung mathematicians was Chu Chih-chieh (fl. 1280–1303), yet we know little about him-, [...] Of greater historical and mathematical interest is the Ssy-yüan yü-chien (Precious Mirror of the Four Elements) of 1303. In the eighteenth century this, too, disappeared in China, only to be rediscovered in the next century. The four elements, called heaven, earth, man, and matter, are the representations of four unknown quantities in the same equation. The book marks the peak in the development of Chinese algebra, for it deals with simultaneous equations and with equations of degrees as high as fourteen. In it the author describes a transformation method that he calls fan fa, the elements of which to have arisen long before in China, but which generally bears the name of Horner, who lived half a millennium later." ^ a b (Boyer 1991, "China and India" p. 205) "A few of the many summations of series found in the Precious Mirror are the following:[...] However, no proofs are given, nor does the topic seem to have been continued again in China until about the nineteenth century. [...] The Precious Mirror opens with a diagram of the arithmetic triangle, inappropriately known in the West as "pascal's triangle." (See illustration.) [...] Chu disclaims credit for the triangle, referring to it as a "diagram of the old method for finding eighth and lower powers". A similar arrangement of coefficients through the sixth power had appeared in the work of Yang Hui, but without the round zero symbol." ^ (Boyer 1991, "Revival and Decline of Greek Mathematics" p. 178) Uncertainty about the life of Diophantus is so great that we do not know definitely in which century he lived. Generally he is assumed to have flourished about 250 CE, but dates a century or more earlier or later are sometimes suggested[...] If this conundrum is historically accurate, Diophantus lived to be eighty-four-years old. [...] The chief Diophantine work known to us is the Arithmetica, a treatise originally in thirteen books, only the first six of which have survived." ^ Oaks, Jeffrey; Christianidis, Jean. The Arithmetica of Diophantus A Complete Translation and Commentary. p. 80. ^ a b c Oaks, Jeffrey; Christianidis, Jean (2013). "Practicing algebra in late antiquity: The problem-solving of Diophantus of Alexandria". Historia Mathematica. 40 (2): 158–160. doi:10.1016/j.hm.2012.09.001. ^ Oaks, Jeffrey; Christianidis, Jean (2013). "Practicing algebra in late antiquity: The problem-solving of Diophantus of Alexandria". Historia Mathematica. 40: 150. ^ Oaks, Jeffrey; Christianidis, Jean (2023). The Arithmetica of Diophantus A Complete Translation and Commentary. pp. 51–52. ^ Oaks, Jeffrey; Christianidis, Jean (2021). The Arithmetica of Diophantus A Complete Translation and Commentary. pp. 53–66. ^ (Boyer 1991, "Revival and Decline of Greek Mathematics" pp. 180–182) "In this respect it can be compared with the great classics of the earlier Alexandrian Age; yet it has practically nothing in common with these or, in fact, with any traditional Greek mathematics. It represents essentially a new branch and makes use of a different approach. Being divorced from geometric methods, it resembles Babylonian algebra to a large extent. But whereas Babylonian mathematicians had been concerned primarily with approximate solutions of determinate equations as far as the third degree, the Arithmetica of Diophantus (such as we have it) is almost entirely devoted to the exact solution of equations, both determinate and indeterminate. [...] Throughout the six surviving books of Arithmetica there is a systematic use of abbreviations for powers of numbers and for relationships and operations. An unknown number is represented by a symbol resembling the Greek letter ζ (perhaps for the last letter of arithmos). [...] It is instead a collection of some 150 problems, all worked out in terms of specific numerical examples, although perhaps generality of method was intended. There is no postulation development, nor is an effort made to find all possible solutions. In the case of quadratic equations with two positive roots, only the larger is give, and negative roots are not recognized. No clear-cut distinction is made between determinate and indeterminate problems, and even for the latter for which the number of solutions generally is unlimited, only a single answer is given. Diophantus solved problems involving several unknown numbers by skillfully expressing all unknown quantities, where possible, in terms of only one of them." ^ a b (Boyer 1991, "Revival and Decline of Greek Mathematics" p. 178) "The chief difference between Diophantine syncopation and the modern algebraic notation is the lack of special symbols for operations and relations, as well as of the exponential notation." ^ a b c (Derbyshire 2006, "The Father of Algebra" pp. 35–36) ^ (Cooke 1997, "Mathematics in the Roman Empire" pp. 167–168) ^ Oaks, Jeffrey; Christianidis, Jean (2023). The Arithmetica of Diophantus A Complete Translation and Commentary. pp. 78–79. There are two major flaws with this trichotomy. First, the language written in books is not always the language in which problems were worked out. In Arabic, problems were often solved in notation on a dust-board or some other temporary surface, and then for inclusion in a book a rhetorical version was composed. Also, because of the two-dimensional character of the Arabic notation, it would have been written and read visually, independent of real or imagined speech. It thus fits nicely into Nesselmann's "symbolic" category. The rhetorical version of the same work, on the other hand, was categorized as being "rhetorical". These two ways of writing algebra do not reflect two stages of the development of algebra but are different ways of expressing the same ideas. Second, Nesselmann was unaware of the conceptual differences between premodern and modern algebra, and thus, he could not have appreciated the leap made in the time of Viète and Descartes that included a radical shift in how notation was interpreted. ^ (Boyer 1991, "Europe in the Middle Ages" p. 257) "The book makes frequent use of the identities [...] which had appeared in Diophantus and had been widely used by the Arabs." ^ (Boyer 1991, "The Mathematics of the Hindus" p. 197) "The oldest surviving documents on Hindu mathematics are copies of works written in the middle of the first millennium B.C., approximately the time during which Thales and Pythagoras lived. [...] from the sixth century B.C." ^ a b (Boyer 1991, "China and India" p. 222) "The Livavanti, like the Vija-Ganita, contains numerous problems dealing with favorite Hindu topics; linear and quadratic equations, both determinate and indeterminate, simple mensuration, arithmetic and geometric progressions, surds, Pythagorean triads, and others." ^ (Boyer 1991, "The Mathematics of the Hindus" p. 207) "He gave more elegant rules for the sum of the squares and cubes of an initial segment of the positive integers. The sixth part of the product of three quantities consisting of the number of terms, the number of terms plus one, and twice the number of terms plus one is the sum of the squares. The square of the sum of the series is the sum of the cubes." ^ (Boyer 1991, "China and India" p. 219) "Brahmagupta (fl. 628), who lived in Central India somewhat more than a century after Aryabhata [...] in the trigonometry of his best-known work, the Brahmasphuta Siddhanta, [...] here we find general solutions of quadratic equations, including two roots even in cases in which one of them is negative." ^ (Boyer 1991, "China and India" p. 220) "Hindu algebra is especially noteworthy in its development of indeterminate analysis, to which Brahmagupta made several contributions. For one thing, in his work we find a rule for the formation of Pythagorean triads expressed in the form m , 1 / 2 (m 2 / n − n) , 1 / 2 (m 2 / n + n) {\displaystyle m,1/2\left(m^{2}/n-n\right),1/2\left(m^{2}/n+n\right)} ; but this is only a modified form of the old Babylonian rule, with which he may have become familiar." ^ a b c d (Boyer 1991, "China and India" p. 221) "he was the first one to give a general solution of the linear Diophantine equation a x + b y = c , {\displaystyle ax+by=c,} where a , b , {\displaystyle a,b,} and c {\displaystyle c} are integers. [...] It is greatly to the credit of Brahmagupta that he gave all integral solutions of the linear Diophantine equation, whereas Diophantus himself had been satisfied to give one particular solution of an indeterminate equation. Inasmuch as Brahmagupta used some of the same examples as Diophantus, we see again the likelihood of Greek influence in India—or the possibility that they both made use of a common source, possibly from Babylonia. It is interesting to note also that the algebra of Brahmagupta, like that of Diophantus, was syncopated. Addition was indicated by juxtaposition, subtraction by placing a dot over the subtrahend, and division by placing the divisor below the dividend, as in our fractional notation but without the bar. The operations of multiplication and evolution (the taking of roots), as well as unknown quantities, were represented by abbreviations of appropriate words. [...] Bhaskara (1114 – c. 1185), the leading mathematician of the twelfth century. It was he who filled some of the gaps in Brahmagupta's work, as by giving a general solution of the Pell equation and by considering the problem of division by zero." ^ a b (Boyer 1991, "China and India" pp. 222–223) "In treating of the circle and the sphere the Lilavati fails also to distinguish between exact and approximate statements. [...] Many of Bhaskara's problems in the Livavati and the Vija-Ganita evidently were derived from earlier Hindu sources; hence, it is no surprise to note that the author is at his best in dealing with indeterminate analysis." ^ a b c (Boyer 1991, "The Arabic Hegemony" p. 227) "The first century of the Muslim empire had been devoid of scientific achievement. This period (from about 650 to 750) had been, in fact, perhaps the nadir in the development of mathematics, for the Arabs had not yet achieved intellectual drive, and concern for learning in other parts of the world had faded. Had it not been for the sudden cultural awakening in Islam during the second half of the eighth century, considerably more of ancient science and mathematics would have been lost. [...] It was during the caliphate of al-Mamun (809–833), however, that the Arabs fully indulged their passion for translation. The caliph is said to have had a dream in which Aristotle appeared, and as a consequence al-Mamun determined to have Arabic versions made of all the Greek works that he could lay his hands on, including Ptolemy's Almagest and a complete version of Euclid's Elements. From the Byzantine Empire, with which the Arabs maintained an uneasy peace, Greek manuscripts were obtained through peace treaties. Al-Mamun established at Baghdad a "House of Wisdom" (Bait al-hikma) comparable to the ancient Museum at Alexandria. Among the faculty members was a mathematician and astronomer, Mohammed ibn-Musa al-Khwarizmi, whose name, like that of Euclid, later was to become a household word in Western Europe. The scholar, who died sometime before 850, wrote more than half a dozen astronomical and mathematical works, of which the earliest were probably based on the Sindhad derived from India." ^ a b (Boyer 1991, "The Arabic Hegemony" p. 234) "but al-Khwarizmi's work had a serious deficiency that had to be removed before it could serve its purpose effectively in the modern world: a symbolic notation had to be developed to replace the rhetorical form. This step the Arabs never took, except for the replacement of number words by number signs. [...] Thabit was the founder of a school of translators, especially from Greek and Syriac, and to him we owe an immense debt for translations into Arabic of works by Euclid, Archimedes, Apollonius, Ptolemy, and Eutocius." ^ a b Gandz and Saloman (1936), The sources of al-Khwarizmi's algebra, Osiris i, pp. 263–277: "In a sense, Khwarizmi is more entitled to be called "the father of algebra" than Diophantus because Khwarizmi is the first to teach algebra in an elementary form and for its own sake, Diophantus is primarily concerned with the theory of numbers". ^ a b (Boyer 1991, "The Arabic Hegemony" p. 230) "Al-Khwarizmi continued: "We have said enough so far as numbers are concerned, about the six types of equations. Now, however, it is necessary that we should demonstrate geometrically the truth of the same problems which we have explained in numbers." The ring of this passage is obviously Greek rather than Babylonian or Indian. There are, therefore, three main schools of thought on the origin of Arabic algebra: one emphasizes Hindu influence, another stresses the Mesopotamian, or Syriac-Persian, tradition, and the third points to Greek inspiration. The truth is probably approached if we combine the three theories." ^ (Boyer 1991, "The Arabic Hegemony" pp. 228–229) "the author's preface in Arabic gave fulsome praise to Mohammed, the prophet, and to al-Mamun, "the Commander of the Faithful."" ^ Corbin, Henry (1998). The Voyage and the Messenger: Iran and Philosophy. North Atlantic Books. p. 44. ISBN 978-1-55643-269-9. Archived from the original on 28 March 2023. Retrieved 19 October 2020. ^ Boyer, Carl B., 1985. A History of Mathematics, p. 252. Princeton University Press. "Diophantus sometimes is called the father of algebra, but this title more appropriately belongs to al-Khowarizmi...", "...the Al-jabr comes closer to the elementary algebra of today than the works of either Diophantus or Brahmagupta..." ^ S Gandz, The sources of al-Khwarizmi's algebra, Osiris, i (1936), 263–277, "Al-Khwarizmi's algebra is regarded as the foundation and cornerstone of the sciences. In a sense, al-Khwarizmi is more entitled to be called "the father of algebra" than Diophantus because al-Khwarizmi is the first to teach algebra in an elementary form and for its own sake, Diophantus is primarily concerned with the theory of numbers." ^ Katz, Victor J. "Stages in the History of Algebra with Implications for Teaching" (PDF). VICTOR J. KATZ, University of the District of Columbia Washington, DC: 190. Archived from the original (PDF) on 27 March 2019. Retrieved 7 October 2017 – via University of the District of Columbia Washington DC, USA. The first true algebra text which is still extant is the work on al-jabr and al-muqabala by Mohammad ibn Musa al-Khwarizmi, written in Baghdad around 825. ^ Esposito, John L. (2000). The Oxford History of Islam. Oxford University Press. p. 188. ISBN 978-0-19-988041-6. Archived from the original on 28 March 2023. Retrieved 29 September 2020. Al-Khwarizmi is often considered the founder of algebra, and his name gave rise to the term algorithm. ^ (Boyer 1991, "The Arabic Hegemony" p. 228) "The Arabs in general loved a good clear argument from premise to conclusion, as well as systematic organization—respects in which neither Diophantus nor the Hindus excelled." ^ a b (Boyer 1991, "The Arabic Hegemony" p. 229) "It is not certain just what the terms al-jabr and muqabalah mean, but the usual interpretation is similar to that implied in the translation above. The word al-jabr presumably meant something like "restoration" or "completion" and seems to refer to the transposition of subtracted terms to the other side of an equation, which is evident in the treatise; the word muqabalah is said to refer to "reduction" or "balancing"—that is, the cancellation of like terms on opposite sides of the equation." ^ a b Rashed, R.; Armstrong, Angela (1994), The Development of Arabic Mathematics, Springer, pp. 11–12, ISBN 978-0-7923-2565-9, OCLC 29181926 ^ a b (Boyer 1991, "The Arabic Hegemony" p. 229) "in six short chapters, of the six types of equations made up from the three kinds of quantities: roots, squares, and numbers (that is x , x 2 , {\displaystyle x,x^{2},} and numbers). Chapter I, in three short paragraphs, covers the case of squares equal to roots, expressed in modern notation as x 2 = 5 x , x 2 / 3 = 4 x , {\displaystyle x^{2}=5x,x^{2}/3=4x,} and 5 x 2 = 10 x , {\displaystyle 5x^{2}=10x,} giving the answers x = 5 , x = 12 , {\displaystyle x=5,x=12,} and x = 2 {\displaystyle x=2} respectively. (The root x = 0 {\displaystyle x=0} was not recognized.) Chapter II covers the case of squares equal to numbers, and Chapter III solves the cases of roots equal to numbers, again with three illustrations per chapter to cover the cases in which the coefficient of the variable term is equal to, more than, or less than one. Chapters IV, V, and VI are more interesting, for they cover in turn the three classical cases of three-term quadratic equations: (1) squares and roots equal to numbers, (2) squares and numbers equal to roots, and (3) roots and numbers equal to squares." ^ (Boyer 1991, "The Arabic Hegemony" pp. 229–230) "The solutions are "cookbook" rules for "completing the square" applied to specific instances. [...] In each case only the positive answer is give. [...] Again only one root is given for the other is negative. [...]The six cases of equations given above exhaust all possibilities for linear and quadratic equations having positive roots." ^ (Boyer 1991, "The Arabic Hegemony" p. 230) "Al-Khwarizmi here calls attention to the fact that what we designate as the discriminant must be positive: "You ought to understand also that when you take the half of the roots in this form of equation and then multiply the half by itself; if that which proceeds or results from the multiplication is less than the units above mentioned as accompanying the square, you have an equation." [...] Once more the steps in completing the square are meticulously indicated, without justification," ^ (Boyer 1991, "The Arabic Hegemony" p. 231) "The Algebra of al-Khwarizmi betrays unmistakable Hellenic elements," ^ (Boyer 1991, "The Arabic Hegemony" p. 233) "A few of al-Khwarizmi's problems give rather clear evidence of Arabic dependence on the Babylonian-Heronian stream of mathematics. One of them presumably was taken directly from Heron, for the figure and dimensions are the same." ^ a b (Boyer 1991, "The Arabic Hegemony" p. 228) "the algebra of al-Khwarizmi is thoroughly rhetorical, with none of the syncopation found in the Greek Arithmetica or in Brahmagupta's work. Even numbers were written out in words rather than symbols! It is quite unlikely that al-Khwarizmi knew of the work of Diophantus, but he must have been familiar with at least the astronomical and computational portions of Brahmagupta; yet neither al-Khwarizmi nor other Arabic scholars made use of syncopation or of negative numbers." ^ a b c d (Boyer 1991, "The Arabic Hegemony" p. 234) "The Algebra of al-Khwarizmi usually is regarded as the first work on the subject, but a recent publication in Turkey raises some questions about this. A manuscript of a work by 'Abd-al-Hamid ibn-Turk, entitled "Logical Necessities in Mixed Equations," was part of a book on Al-jabr wa'l muqabalah which was evidently very much the same as that by al-Khwarizmi and was published at about the same time—possibly even earlier. The surviving chapters on "Logical Necessities" give precisely the same type of geometric demonstration as al-Khwarizmi's Algebra and in one case the same illustrative example x 2 + 21 = 10 x . {\displaystyle x^{2}+21=10x.} . In one respect 'Abd-al-Hamad's exposition is more thorough than that of al-Khwarizmi for he gives geometric figures to prove that if the discriminant is negative, a quadratic equation has no solution. Similarities in the works of the two men and the systematic organization found in them seem to indicate that algebra in their day was not so recent a development as has usually been assumed. When textbooks with a conventional and well-ordered exposition appear simultaneously, a subject is likely to be considerably beyond the formative stage. [...] Note the omission of Diophantus and Pappus, authors who evidently were not at first known in Arabia, although the Diophantine Arithmetica became familiar before the end of the tenth century." ^ a b (Derbyshire 2006, "The Father of Algebra" p. 49) ^ O'Connor, John J.; Robertson, Edmund F. (1999), "Arabic mathematics: forgotten brilliance?", MacTutor History of Mathematics Archive, University of St Andrews "Algebra was a unifying theory which allowed rational numbers, irrational numbers, geometrical magnitudes, etc., to all be treated as "algebraic objects"." ^ Jacques Sesiano, "Islamic mathematics", p. 148, in Selin, Helaine; D'Ambrosio, Ubiratan, eds. (2000), Mathematics Across Cultures: The History of Non-Western Mathematics, Springer, ISBN 978-1-4020-0260-1 ^ a b Berggren, J. Lennart (2007). "Mathematics in Medieval Islam". The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook. Princeton University Press. p. 518. ISBN 978-0-691-11485-9. ^ a b (Boyer 1991, "The Arabic Hegemony" p. 239) "Abu'l Wefa was a capable algebraist as well as a trionometer. [...] His successor al-Karkhi evidently used this translation to become an Arabic disciple of Diophantus—but without Diophantine analysis! [...] In particular, to al-Karkhi is attributed the first numerical solution of equations of the form a x 2 n + b x n = c {\displaystyle ax^{2n}+bx^{n}=c} (only equations with positive roots were considered)," ^ O'Connor, John J.; Robertson, Edmund F., "Abu Bekr ibn Muhammad ibn al-Husayn Al-Karaji", MacTutor History of Mathematics Archive, University of St Andrews ^ a b c d e (Boyer 1991, "The Arabic Hegemony" pp. 241–242) "Omar Khayyam (c. 1050 – 1123), the "tent-maker", wrote an Algebra that went beyond that of al-Khwarizmi to include equations of third degree. Like his Arab predecessors, Omar Khayyam provided for quadratic equations both arithmetic and geometric solutions; for general cubic equations, he believed (mistakenly, as the sixteenth century later showed), arithmetic solutions were impossible; hence he gave only geometric solutions. The scheme of using intersecting conics to solve cubics had been used earlier by Menaechmus, Archimedes, and Alhazan, but Omar Khayyam took the praiseworthy step of generalizing the method to cover all third-degree equations (having positive roots). .. For equations of higher degree than three, Omar Khayyam evidently did not envision similar geometric methods, for space does not contain more than three dimensions, [...] One of the most fruitful contributions of Arabic eclecticism was the tendency to close the gap between numerical and geometric algebra. The decisive step in this direction came much later with Descartes, but Omar Khayyam was moving in this direction when he wrote, "Whoever thinks algebra is a trick in obtaining unknowns has thought it in vain. No attention should be paid to the fact that algebra and geometry are different in appearance. Algebras are geometric facts which are proved."" ^ O'Connor, John J.; Robertson, Edmund F., "Sharaf al-Din al-Muzaffar al-Tusi", MacTutor History of Mathematics Archive, University of St Andrews ^ Rashed, Roshdi; Armstrong, Angela (1994), The Development of Arabic Mathematics, Springer, pp. 342–343, ISBN 978-0-7923-2565-9 ^ Berggren, J. L. (1990). "Innovation and Tradition in Sharaf al-Din al-Tusi's Muadalat". Journal of the American Oriental Society. 110 (2): 304–309. doi:10.2307/604533. JSTOR 604533. Rashed has argued that Sharaf al-Din discovered the derivative of cubic polynomials and realized its significance for investigating conditions under which cubic equations were solvable; however, other scholars have suggested quite difference explanations of Sharaf al-Din's thinking, which connect it with mathematics found in Euclid or Archimedes. ^ Nasehpour, Peyman (2018). "A Brief History of Algebra with a Focus on the Distributive Law and Semiring Theory". arXiv:1807.11704 [math.HO]. ^ Victor J. Katz, Bill Barton (October 2007), "Stages in the History of Algebra with Implications for Teaching", Educational Studies in Mathematics, 66 (2): 185–201 [192], doi:10.1007/s10649-006-9023-7, S2CID 120363574 ^ Tjalling J. Ypma (1995), "Historical development of the Newton-Raphson method", SIAM Review 37 (4): 531–551, doi:10.1137/1037125 ^ "Fibonacci's 'Numbers': The Man Behind The Math". NPR. ^ a b c O'Connor, John J.; Robertson, Edmund F., "Abu'l Hasan ibn Ali al Qalasadi", MacTutor History of Mathematics Archive, University of St Andrews ^ (Boyer 1991, "Euclid of Alexandria pp. 192–193) "The death of Boethius may be taken to mark the end of ancient mathematics in the Western Roman Empire, as the death of Hypatia had marked the close of Alexandria as a mathematical center; but work continued for a few years longer at Athens. [...] When in 527 Justinian became emperor in the East, he evidently felt that the pagan learning of the Academy and other philosophical schools at Athens was a threat to orthodox Christianity; hence, in 529 the philosophical schools were closed and the scholars dispersed. Rome at the time was scarcely a very hospitable home for scholars, and Simplicius and some of the other philosophers looked to the East for haven. This they found in Persia, where under King Chosroes they established what might be called the "Athenian Academy in Exile."(Sarton 1952; p. 400)." ^ E.g. Bashmakova & Smirnova (2000:78), Boyer (1991:180), Burton (1995:319), Derbyshire (2006:93), Katz & Parshall (2014:238), Sesiano (1999:125), and Swetz (2013:110) ^ Descartes (1637:301–303) ^ Descartes (1925:9–14) ^ Cajori (1919:698); Cajori (1928:381–382) ^ Eneström (1905:317) ^ E.g. Tropfke (1902:150). But Gustaf Eneström (1905:316–317) showed that Descartes, in a letter written in 1619, used the German symbol in clear contrast to his own x . {\displaystyle x.} ^ A crossed numeral 1 was used by Pietro Cataldi for the first power of the unknown. The link between this convention and x {\displaystyle x} is attributed by Cajori to Gustav Wertheim, but Cajori (1919:699; 1928:382) finds no evidence to support it. ^ Cajori (1919:699) ^ See, for example, the TED talk by Terry Moore, entitled "Why Is 'x' the Unknown?", released in 2012. ^ Alcalá (1505) ^ Lagarde (1884). ^ Jacob (1903:519). ^ Rider (1982) lists five treatises on algebra published in Spanish in the sixteenth century, all of which use "cosa": Aurel (1552), Ortega (1552), Díez (1556), Pérez de Moya (1562), and Nunes (1567). The latter two works also abbreviate cosa as "co."—as does Puig (1672). ^ The forms are absent from Alonso (1986), Kasten & Cody (2001), Oelschläger (1940), the Spanish Royal Academy's online diachronic corpus of Spanish (CORDE), and Davies's Corpus del Español. ^ "Why x?". Retrieved 2019-05-30. ^ Struik (1969), 367.[full citation needed] ^ Andrew Warwick (2003) Masters of Theory: Cambridge and the Rise of Mathematical Physics, Chicago: University of Chicago Press ISBN 0-226-87374-9 ^ a b c d (Boyer 1991, "The Arabic Hegemony" p. 228) "Diophantus sometimes is called "the father of algebra", but this title more appropriately belongs to Abu Abdullah bin mirsmi al-Khwarizmi. It is true that in two respects the work of al-Khwarizmi represented a retrogression from that of Diophantus. First, it is on a far more elementary level than that found in the Diophantine problems and, second, the algebra of al-Khwarizmi is thoroughly rhetorical, with none of the syncopation found in the Greek Arithmetica or in Brahmagupta's work. Even numbers were written out in words rather than symbols! It is quite unlikely that al-Khwarizmi knew of the work of Diophantus, but he must have been familiar with at least the astronomical and computational portions of Brahmagupta; yet neither al-Khwarizmi nor other Arabic scholars made use of syncopation or of negative numbers." ^ Herscovics, Nicolas; Linchevski, Liora (1 July 1994). "A cognitive gap between arithmetic and algebra". Educational Studies in Mathematics. 27 (1): 59–78. doi:10.1007/BF01284528. ISSN 1573-0816. S2CID 119624121. This would have come as a surprise to al-Khwarizmi, considered to be the father of algebra (Boyer/Merzbach, 1991), who introduced it to the Mediterranean world around the ninth century ^ Dodge, Yadolah (2008). The Concise Encyclopedia of Statistics. Springer Science & Business Media. p. 1. ISBN 9780387317427. The term algorithm comes from the Latin pronunciation of the name of the ninth century mathematician al-Khwarizmi, who lived in Baghdad and was the father of algebra. ^ (Derbyshire 2006, "The Father of Algebra" p. 31) "Van der Waerden pushes the parentage of algebra to a point later in time, beginning with the mathematician al-Khwarizmi" ^ (Derbyshire 2006, "The Father of Algebra" p. 31) "Diophantus, the father of algebra, in whose honor I have named this chapter, lived in Alexandria, in Roman Egypt, in either the 1st, the 2nd, or the 3rd century CE." ^ J. Sesiano, K. Vogel, "Diophantus", Dictionary of Scientific Biography (New York, 1970–1990), "Diophantus was not, as he has often been called, the father of algebra." ^ (Derbyshire 2006, "The Father of Algebra" p. 31) "Kurt Vogel, for example, writing in the Dictionary of Scientific Biography, regards Diophantaus's work as not much more algebraic than that of the old Babylonians" ^ (Boyer 1991, "The Arabic Hegemony" p. 230) "The six cases of equations given above exhaust all possibilities for linear and quadratic equations having positive root. So systematic and exhaustive was al-Khwarizmi's exposition that his readers must have had little difficulty in mastering the solutions." ^ Katz, Victor J. (2006). "STAGES IN THE HISTORY OF ALGEBRA WITH IMPLICATIONS FOR TEACHING" (PDF). VICTOR J.KATZ, University of the District of Columbia Washington DC: 190. Archived from the original (PDF) on 2019-03-27. Retrieved 2019-08-06 – via University of the District of Columbia Washington DC, USA. The first true algebra text which is still extant is the work on al-jabr and al-muqabala by Mohammad ibn Musa al-Khwarizmi, written in Baghdad around 825. ^ Oaks, Jeffrey (2014). The Oxford Encyclopedia of Islam and Philosophy, Science, and Technology. p. 458. Judgments by historians that either downplay al-Khwārizmī's importance because of his lack of originality or force on him the title "inventor of the science of algebra" presuppose a modern, Western ideal of mathematical achievement that is not applicable to our ninth-century scholar. Al-Khwārizmī's fame in the Islamic world rests on his success in writing books that served a foundational role for practical study and for future scholarship. ^ Christianidis, Jean (2007). "The way of Diophantus: Some clarifications on Diophantus' method of solution". Historia Mathematica. 34 (3): 303. doi:10.1016/j.hm.2006.10.003.

Alcalá, Pedro de (1505), De lingua arabica, Granada Edition by Paul de Lagarde, Göttingen: Arnold Hoyer, 1883 Alonso, Martín [in Spanish] (1986), Diccionario del español medieval, Salamanca: Universidad Pontificia de Salamanca Aurel, Marco (1552), Libro primero de arithmetica algebratica, Valencia: Joan de Mey Bashmakova, I; Smirnova, G. (2000). The Beginnings and Evolution of Algebra. Dolciani Mathematical Expositions. Vol. 23. Translated by Abe Shenitzer. The Mathematical Association of America. Boyer, Carl B. (1991), A History of Mathematics (2nd ed.), John Wiley & Sons, Inc., ISBN 978-0-471-54397-8 Burton, David M. (1995), Burton's History of Mathematics: An Introduction (3rd ed.), Dubuque: Wm. C. Brown Burton, David M. (1997), The History of Mathematics: An Introduction (3rd ed.), The McGraw-Hill Companies, Inc., ISBN 978-0-07-009465-9 Cajori, Florian (1919), "How x Came to Stand for Unknown Quantity", School Science and Mathematics, 19 (8): 698–699, doi:10.1111/j.1949-8594.1919.tb07713.x Cajori, Florian (1928), A History of Mathematical Notations, Chicago: Open Court Publishing, ISBN 9780486161167 {{citation}}: ISBN / Date incompatibility (help) Cooke, Roger (1997), The History of Mathematics: A Brief Course, Wiley-Interscience, ISBN 978-0-471-18082-1 Derbyshire, John (2006), Unknown Quantity: A Real And Imaginary History of Algebra, Washington, DC: Joseph Henry Press, ISBN 978-0-309-09657-7 Descartes, René (1637), La Géométrie, Leyde: Ian Maire. Online 2008 ed. by L. Hermann, Project Gutenberg. Descartes, René (1925), The Geometry of René Descartes, Chicago: Open Court, ISBN 9781602066922 {{citation}}: ISBN / Date incompatibility (help) Díez, Juan (1556), Sumario compendioso de las quentas de plata y oro que en los reynos del Piru son necessarias a los mercaderes: y todo genero de tratantes, con algunas reglas tocantes al arithmetica, Mexico City Eneström, Gustaf (1905), "Kleine Mitteilungen", Bibliotheca Mathematica, Ser. 3, 6 (online access only in U.S.) Flegg, Graham (1983), Numbers: Their History and Meaning, Dover publications, ISBN 978-0-486-42165-0 Heath, Thomas Little (1981a), A History of Greek Mathematics, Volume I, Dover publications, ISBN 978-0-486-24073-2 Heath, Thomas Little (1981b), A History of Greek Mathematics, Volume II, Dover publications, ISBN 978-0-486-24074-9 Jacob, Georg (1903), "Oriental Elements of Culture in the Occident", Annual Report of the Board of Regents of the Smithsonian Institution [...] for the Year Ending June 30, 1902: 509–529 Kasten, Lloyd A.; Cody, Florian J. (2001), Tentative Dictionary of Medieval Spanish (2nd ed.), New York: Hispanic Seminary of Medieval Studies Katz, Victor J.; Parshall, Karen Hunger (2014), Taming the Unknown: A History of Algebra from Antiquity to the Early Twentieth Century, Princeton, NJ: Princeton University Press, ISBN 978-1-400-85052-5 Lagarde, Paul de (1884), "Woher stammt das x der Mathematiker?", Mittheilungen, vol. 1, Goettingen: Dieterichsche Sortimentsbuchhandlung, pp. 134–137 Nunes, Pedro (1567), Libro de algebra en arithmetica y geometria, Antwerp: Arnoldo Birckman Oelschläger, Victor R. B. (1940), A Medieval Spanish Word-List, Madison: University of Wisconsin Press Ortega, Juan de (1552), Tractado subtilissimo de arismetica y geometria, Granada: René Rabut Pérez de Moya, Juan [in Spanish] (1562), Aritmética práctica y especulativa, Salamanca: Mathias Gast Puig, Andrés (1672), Arithmetica especulativa y practica; y arte de algebra, Barcelona: Antonio Lacavalleria Rider, Robin E. (1982), A Bibliography of Early Modern Algebra, 1500–1800, Berkeley: Berkeley Papers in History of Science Sesiano, Jacques (1999), An Introduction to the History of Algebra: Solving Equations from Mesopotamian Times to the Renaissance, Providence, RI: American Mathematical Society, ISBN 9780821844731 Stillwell, John (2004), Mathematics and its History (2nd ed.), Springer Science + Business Media Inc., ISBN 978-0-387-95336-6 Swetz, Frank J. (2013), The European Mathematical Awakening: A Journey Through the History of Mathematics, 1000–1800 (2nd ed.), Mineola, NY: Dover Publications, ISBN 9780486498058 Tropfke, Johannes (1902), Geschichte der Elementar-Mathematik in systematischer Darstellung, vol. 1, Leipzig: Von Veit & Comp. Wikiquote has quotations related to History of algebra. "Commentary by Islam's Sheikh Zakariyya al-Ansari on Ibn al-Hā'im's Poem on the Science of Algebra and Balancing Called the Creator's Epiphany in Explaining the Cogent" featuring the basic concepts of algebra dating back to the 15th century, from the World Digital Library. Retrieved from "